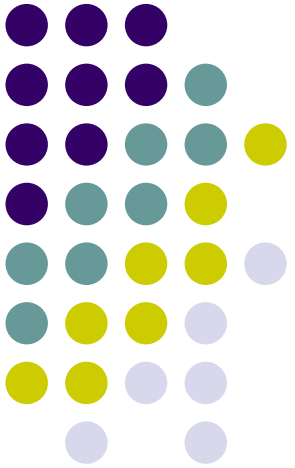
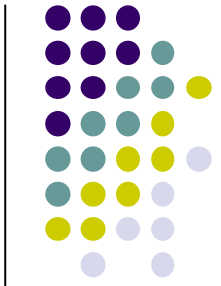
計算組合數 Comb(n,k)

丁培毅



計算 C_k^n and P_k^n



- 版本一 **long int**

$$\frac{n!}{k!\,(n-k)!}$$

```
01 #include <stdio.h>
02 #include <stdlib.h>
03 int factorial(int);
04 int find_nCk(int, int);
05 int find_nPk(int, int);
06 int main(void) {
07     int n, k, nCk, nPk;
08     printf("Enter the value of n and k\n");
09     scanf("%d%d",&n,&k);
10     nCk = find_nCk(n, k);
11     nPk = find_nPk(n, k);
12     printf("%dC%d = %d\n", n, k, nCk);
13     printf("%dP%d = %d\n", n, k, nPk);
14     system("pause");
15     return 0;
16 }
```

```
17 int find_nCk(int n, int k) {
18     return factorial(n) / (factorial(k)*factorial(n-k));
19 }
```

$$\frac{n!}{(n-k)!}$$

```
20 int find_nPk(int n, int k) {
21     return factorial(n)/factorial(n-k);
22 }
```

```
23 int factorial(int n) {
24     int i, result=1;
25     for (i=1; i<=n; i++)
26         result = result*i;
27     return result;
28 }
```

這樣子寫理論上是對的,
實際上卻有很嚴重的問題!!

$$P_3^{15} = 15 \cdot 14 \cdot 13 = 2730$$

$15! = 1307674368000$

$2^{31}-1 = 2147473647$

計算 C_k^n and P_k^n

- 版本二 **long long int**

```
01 #include <stdio.h>
02 #include <stdlib.h>
03 long long int factorial(int);
04 int find_nCk(int, int);
05 int find_nPk(int, int);
06 int main(void) {
07     int n, k, nCk, nPk;
08     printf("Enter the value of n and k\n");
09     scanf("%d%d",&n,&k);
10     nCk = find_nCk(n, k);
11     nPk = find_nPk(n, k);
12     printf("%dC%d = %d\n", n, k, nCk);
13     printf("%dP%d = %d\n", n, k, nPk);
14     system("pause");
15     return 0;
16 }
```

```
17 int find_nCk(int n, int k) {
18    return factorial(n)/(factorial(k)*factorial(n-k));
19 }
```

```
20 int find_nPk(int n, int k) {
21    return factorial(n)/factorial(n-k);
22 }
```

```
23 long long int factorial(int n) {
24    int i; long long int result=1;
25    for (i=1; i<=n; i++)
26       result = result*i;
27    return result;
28 }
```

$$P_3^{21} = 21 \cdot 20 \cdot 19 = 7980$$

好一些，但是仍然有很大的限制

$21! = 51090942171709440000$

$2^{63}-1 = 9223372036854775807$

計算 C_k^n and P_k^n

- 版本三　**skip unnecessary factorial　n!, (n-k)!**

$$P_k^n = n\,(n-1)\cdots(n-k+1)$$

```
int find_nPk(int n, int k) {
    int i, result=1;
    for (i=0; i<k; i++)
        result = result * (n-i);
    return result;
}
```

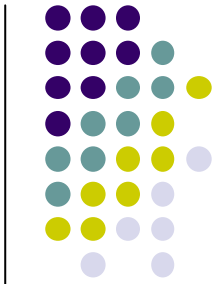
Ex. 30P6 = 427518000
30P7 = 1670497408
30P8 = -233265280
100P5 = 444567808
100P6 = -715731200

```
long long int find_nPk(int n, int k) {
    int i;
    long long int result=1;
    for (i=0; i<k; i++)
        result = result * (n-i);
    return result;
}
```

Ex. 30P13 = 74574707695488000
30P14 = -576904376547659161 6
100P10 = 747541873440081715 2
100P11 = 870489944252968550 4
100P12 = -272007106591580 16

```
printf("result = %I64d\n", result);
printf("result = %lld\n", result);
```

計算 C_k^n and P_k^n



- 版本三 **(cont'd)**

```
long long find_nCk(int n, int k)  {
    int i;
    long long int numerator=1, denominator=1;
    for (i=0; i<k; i++) {
        numerator *= n-i;
        denominator *= k-i;
    }
    return numerator/denominator;
}
```

30C14 = -66175233
100C10 = 2060025003968

```
long long int find_nCk(int n, int k) {
    int i;
    double result=1.0;
    for (i=0; i<k; i++)
        result *= ((double)(n-i))/(k-i);
    return (long long int) (result+0.5);
}
```

100C18 = -9223372036854775808

$$C_k^n = \frac{n\,(n-1)\cdots(n-k+1)}{k\,(k-1)\cdots 1}$$

```
long long find_nCk(int n, int k) {
    int i;
    long long numerator=1;
    long long denominator=1;
    for (i=0; i<k; i++) {
        numerator *= n-i;
        if (numerator % (k-i) == 0)
            numerator /= k-i;
        else
            denominator *= k-i;
    }
    return numerator/denominator;
}
```

100C15 = 332765423377859

程式該怎樣寫才能計算出 C_{33}^{67}

- Comb(66,33) = 7,219,428,434,016,265,740
  2^{63}-1 = 9,223,372,036,854,775,807
  Comb(67,33) = 14,226,520,737,620,288,370
  2^{64}-1 = 18,446,744,073709,551,616
- 由前面版本三的程式你可以看到直接讓分子乘分子、分母乘分母的話很快就爆表了, 第二個程式只是把正下方可以整除的分母除掉就可以看到改善了 (沒有試看看其它的分母, 也沒有沒有嘗試把公因數除掉)
- 因為 Comb(n,k) 一定是整數, 所有的分母拆解成質因數乘積一定可以和分子的質因數乘積消掉, 所以基本的方法就是記住每一個分子和分母, 每一個分母都和所有的分子找公因數, 除掉公因數

計算 C_{33}^{67}

- 需要計算**最大公因數**的程式

```
01 int gcd(int p, int q)  { // assume p >= q
02     if (p % q == 0)
03         return q;
04     else
05         return gcd(q, p % q);
06 }
```

- 動手計算看看

C_5^{10}

$$\frac{2 \quad 9 \quad 2 \quad 7 \quad 1}{10 \quad 9 \quad 8 \quad 7 \quad 6} = \frac{5 \quad 4 \quad 3 \quad 2 \quad 1}{} = 252$$

- 要像上面這樣子算的話, 程式需要記住所有部份消去的分子, 分母一定都會被消完

記憶體是很好用的東西

- 可以設計一個陣列紀錄所有的分子 10, 9, 8, 7, 6, 然後每一個分母denum和所有的分子num[j]一個一個找最大公因數d來消去, 直到每一個分母都完全消去成為1 (這個條件不需要檢查, 只需要把所有分子試完)

```
int n=10, k=5, i, j, d, denum, nCk;
int num[67] = {10, 9, 8, 7, 6}; // 先暴力放進去
for (i=k; i>1; i--)
    for (denum=i,j=0; denum>1&&j<k; j++)
        if ((num[j]>1) && ((d=gcd(denum,num[j]))>1))
            denum/=d, num[j]/=d;  // 消去 d
for (nCk=1,i=0; i<k; i++) nCk *= num[i];
printf("nCk = %d\n", nCk);
```

如果不允許使用額外的記憶體

- 版本三的第二個程式就是沒有用陣列來紀錄分子, 也沒有考慮分母是可以拆開成為質因數乘積, 在不同的分子裡找到可以消去的質因數
- 如果不使用額外記憶體, 要稍微改進一下的話, 第一就是numerator不需要整除k-i, 只需要把最大公因數 d=gcd(numerator*(n-i), denumenator*(k-i))消去就可以, numerator 和denumenator兩個變數裡累積沒有完全消去的因數, 不過這樣子計算時, 還是有可能在計算中間數值 numerator*(n-i) 或是 denumenator*(k-i) 時比 Comb(n,k) 提早產生溢位

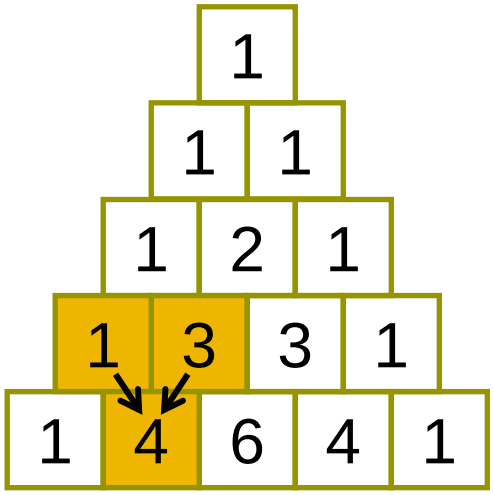
如果允許使用更多的記憶體

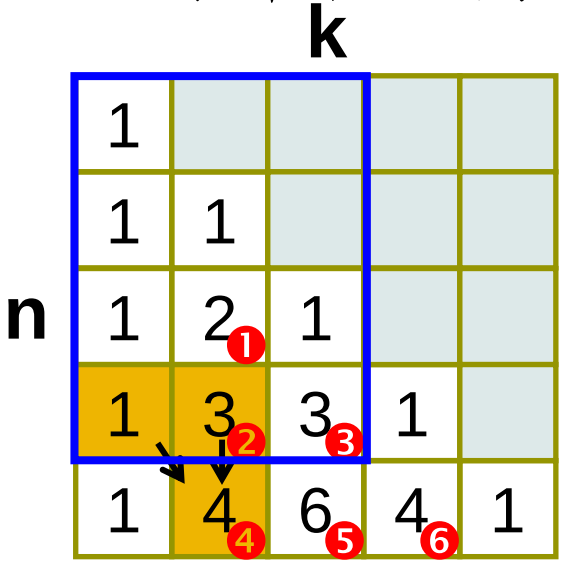
- 我們可以運用遞推公式來計算 Comb(67,33)

  Comb(n,k) = Comb(n-1,k-1) + Comb(n-1,k)

- 這種計算需要清楚地掌握資料陣列的配置方法以及資料計算的順序, 下圖左是一般的帕斯卡三角形

- 下圖右我們用二維的陣列來放Comb(n,k), n 是橫列, k 是直行, k==0和n==k時固定都是1, 其它數字計算的順序標示如下





如果只需要計算 **Comb(n*,k*)** 只須計算小於等於 **n*, k*** 的就足夠了

帕斯卡三角形

- 假設我們定義 long long comb[67][67]; 的二維陣列
  1. 先把固定的數值填好: **comb[*][0], comb[i][i]**

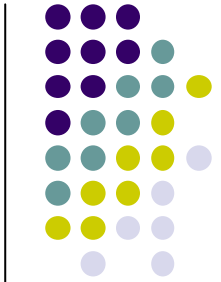
```
for (i=0; i<=n; i++) comb[i][0] = comb[i][i] = 1;
```

  2. 由第三列開始一列一列往下填

```
for (i=2; i<=n; i++)
    for (j=1; j<i && j<=k; j++)
        comb[i][j] = comb[i-1][j-1] + comb[i-1][j];
```

  - 如果使用的記憶體數量是很重要的考量, 這個程式也可以只使用兩列的記憶體, long long comb[2][67]; 由前一列的組合數算本列的組合數, 兩列的記憶體輪流切換
  - 請注意程式多使用記憶體以後演算法變得非常簡潔

偵測「溢位」- 純軟體



```
#define INT64_MIN_VALUE (0x8000000000000000LL)
#define INT64_MAX_VALUE (0x7FFFFFFFFFFFFFFFLL)
int isMultOverflow(long long a, long long b) {
    long long ap, bp, cp, max=INT64_MAX_VALUE;
    if (a==INT64_MIN_VALUE) return b!=1;
    if (b==INT64_MIN_VALUE) return a!=1;
    ap = a >= 0 ? a : -a;
    bp = b >= 0 ? b : -b;
    cp = max / bp;
    return (ap > cp);
}
```

計算每一個乘法之前, 先判斷一下是否會發生溢位, 不會的話才乘起來

等於是花兩倍的時間
像是計算 Comb(500, n) 真的不知道 64 位元可以算到多大的 n 值, 只好邊算邊測試

偵測「溢位」- 硬體狀態旗標

- 主要運用CPU的指令 **JNO (jump no overflow)**

gcc/g++

```
int isOverflow()
{
    __asm__ ("movl  $0, %eax;");
    __asm__ ("jno   end;");
    __asm__ ("movl  $1, %eax;");
    __asm__ ("end:");
}
```

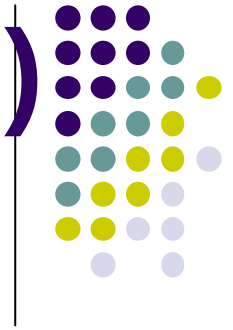
如果偵測位置離開運算敘述太遠就會沒有效果

Visiual C/C++

```
int isOverflow()
{
    _asm mov   eax, 0
    _asm jno   end
    _asm mov   eax, 1
    _asm end:
}
```

vc2010 把64位元乘法用一個函式來完成, 這種偵測就完全無效

C++ Exception Handling (1/3)

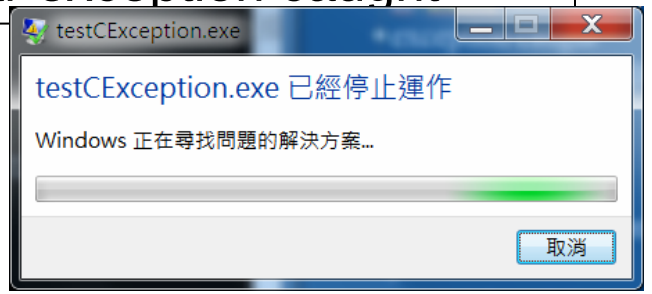


- There are three exception handling mechanisms you can use with C++ on MS Windows:
  - **C++ exceptions**, implemented by the compiler (**try/catch**), cl /EHsc
  - **Structured Exception Handling (SEH),** provided by Windows (**__try** / **__except**), cl /EHa
  - **MFC exception macros** (TRY, CATCH - built on top of SEH / C++ exceptions - see also TheUndeadFish's comment)
- C++ exceptions guarantee **automatic cleanup** during stack unwinding (i.e. dtors of local objects), other mechanisms don't.
- C++ exceptions only occur when they are **explicitly thrown**. Structured Exceptions may occur for any operations (especially **divide by zero**, H/W, and invalid memory access, S/W).
- MFC introduces the macros to support exception handling even if compilers didn't implement them.

C++ Exception Handling (2/3)

```
for (y=3; y>=0; y--)
   try {
        int x=1, *z=0;
        if (y==3) throw exception("explicit");
        else if (y==2) throw 'a';
        else if (y==1) *z = 10;
        else
            x = x/y;
        cout << "end\n";
    }
    catch (exception) {
        cout << "exception caught\n";
    }
    catch (char e) {
        cout << "char exception caught\n";
    }
    catch (...) {
        cout << "... caught\n";
   }
```

```
cl /EHsc
testCException.cpp
exception caught
char exception caught
```




```
cl /EHa testCException.cpp
exception caught
char exception caught
... caught
... caught
```

C++ Exception Handling (3/3)

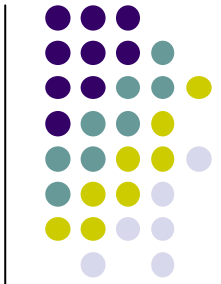
```
for (y=3; y>=0; y--)
    __try {
      int x=1, *z=0;
      if (y==3)
         RaiseException(15,0,0,NULL);
      else if (y==2) throw 'a';
      else if (y==1) *z = 10;
      else
         x = x/y;
      cout << "end\n";bo
    }
    __except (FilterFunction(GetExceptionCode())) {
       cout << GetExceptionCode() << " exception caught\n";
    }
```

```
DWORD FilterFunction(unsigned int code) {
    if (code == 15)
       return EXCEPTION_CONTINUE_SEARCH;
    else
       return EXCEPTION_EXECUTE_HANDLER;
}
```

Something like rethrow

Both cl /EHsc or cl /EHa work.

更大的組合數 Comb(500,250)



- 需要使用大整數類別
- 來運算
- 例如 Crypto++, GMP, Java

```
Integer result;
result = 1;
for (i=0; i<n; i++) {
    d = gcd(m-i, i+1);
    num = (m-i) / d;
    denum = (i+1) / d;
    result /= denum;
    result *= num;
}
```

此程式輸入 m >= n >= 0, 輸出組合數 comb(m, n):
請輸入 m: 500
請輸入 n: 250
comb(500,250)=
1167443157882776829209347347621766196592300811803114461241002849578111126736084737156664177755216053768108659027099895801600374682263939000427968722 56.