RSA Cryptosystem

密碼學與應用

海洋大學資訊工程系

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m * 8 is the ciphertext
$\mathbf{8^{-1}}$ is the private key (if nobody can derive this from the public key, then this system is secure)

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  3. Every possible subset sum is unique

     pf: given x, assume $x = \sum_{j \in S} a_j = \sum_{j \in T} a_j$, where S $\neq$ T, assume $\max_{j \in S}\{a_j\}$ $\max_{j \in T}\{a_j\}$ ….

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- choose a number ***b*** in Z_p^*, e.g. $p = 101$, ***b*** $= 23$, and convert the **super-increasing sequence** to a **normal knapsack sequence**

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- **G**iven a number d, finding a subset $\{b_j\} \subseteq$B s.t.

  $$d = \sum_j b_j \pmod{p}$$

  is an NP-complete problem, e.g. 94 = 11 + 14 + 69

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  - recovered message is $(111100)_2 = (60)_{10}$

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both functions are candidates for **trapdoor one way function**

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  $\phi(n) = 10 \cdot 12 = 120$, $e^{-1} = 103$

Trouble: How do we know $\phi(n)$?

- Solving square root of y modulo n is difficult!!!

  y x^2 (mod n)

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  e.g. n = 11 · 13 = 143

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RSA and Rabin Function

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Remember solving square root of y modulo a prime number p is very easy

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by computing the modular exponentiation

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$\Rightarrow (x_1^e)^d \equiv (x_2^e)^d \pmod n$

$\Rightarrow (x_1)^{1+k\,\phi(n)} \equiv (x_2)^{1+k\,\phi(n)} \pmod n$

$\Rightarrow x_1 \equiv x_2 \pmod n$

RSA Function is a Permutation

- RSA function is a permutation: (1-1 and onto, bijective)
- Goal: "$\forall x_1, x_2 \in Z_n$ if $x_1^e \equiv x_2^e \pmod n$ then $x_1 = x_2$"

$\forall x \neq r \cdot p,\ x^{p-1} \equiv 1 \pmod p,\ \forall x \neq s \cdot q,\ x^{q-1} \equiv 1 \pmod q$

$\Rightarrow \forall k, \forall x \neq r \cdot p,\ x^{k\phi(n)} \equiv 1 \pmod p, \forall x \neq s \cdot q,\ x^{k\phi(n)} \equiv 1 \pmod q$

CRT $\Rightarrow \forall k, \forall x,\ x^{k\phi(n)+1} \equiv x \pmod p,\ x^{k\phi(n)+1} \equiv x \pmod q$

$\Rightarrow \forall k, \forall x,\ x^{k\phi(n)+1} \equiv x \pmod n$

- $\gcd(e, \phi(n)) = 1 \Rightarrow$ inverse of e (mod $\phi(n)$) exists
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Note: Euler Thm is valid only when $x \in Z_n^*$

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- Bob computes the four square roots $\pm m_1, \pm m_2$ using CRT, one of them satisfying the fixed message format is the recovered message

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  - In order to let the Rabin function have inverse, it is necessary to make the Rabin function a permutation, ie. 1-1 and onto. Therefore, the number of elements in the domain of the Rabin function should also be (p-1)(q-1)/4 for n=p·q. There are 4 possible numbers with their square equal to y, and we have to make 3 of them illegal.

Number of Quadratic Residues

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  pf: find a primitive g, at least $\{g^2, g^4, \ldots g^{p-1}\}$ are QR_p's

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- For a composite modulus p·q: number of QR_n's in $Z_{p\cdot q}^*$ is (p-1)(q-1)/4

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  assume there are (p-1)(q-1)/4+1 QR_n's in Z_n^*, since there are four square roots of a QR modulo p·q, these QR_n's have (p-1)(q-1)+4 square roots in total. There must be some repeated elements in this QR_n, therefore, there are at most (p-1)(q-1)/4 QR_n's in Z_n^*

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- maple('m4:= chrem([-r1, -r2], [p, q])') % 15792521272200376029628 22

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will be illustrated later after factorization

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  - ⋆ $\Rightarrow$
    - given a quadratic residue y if you can find the four square roots $\pm x_1$ and $\pm x_2$ for y in polynomial time
    - you can factor n by trying $\gcd(x_1-x_2, n)$ and $\gcd(x_1+x_2, n)$

Basic Factoring Principle (1/4)

- Let n be an integer and suppose there exist integers x and y with $x^2 \equiv y^2 \pmod{n}$, but $x \neq \pm y \pmod{n}$. Then ❶ n is composite, ❷ both gcd(x-y, n) and gcd(x+y, n) are nontrivial factors of n.

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  Case 2: assume d is 1 (the trivial factor)

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- Let n be an integer and suppose there exist integers x and y with $x^2 \equiv y^2 \pmod{n}$, but $x \neq \pm y \pmod{n}$. Then ❶ n is composite, ❷ both gcd(x-y, n) and gcd(x+y, n) are nontrivial factors of n.

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  let d = gcd(x-y, n).

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- $x^2 \equiv y^2 \pmod{p}$ implies $x \equiv \pm y \pmod{p}$ since $p \mid (x+y)(x-y)$ implies $p \mid (x+y)$ or $p \mid (x-y)$,

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    - $x \equiv -y \pmod{p}$ and $x \equiv y \pmod{q}$ $\Rightarrow$ $x \equiv -z \pmod{n}$
  - as long as we have z (where $z \not\equiv \pm y$), we can factor n into $\gcd(y-z, n)$ and $\gcd(y+z, n)$

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- Compute $b_k \equiv b_{k-1}^2 \pmod{n}$
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n will pass Fermat test
n is called pseudo prime
with respect to base a

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④ ①, ②, and ③ are not true,
$b_k \equiv a^{n-1} \pmod n$
if $b_k \neq 1 \pmod n$ n is **composite**
since if n is prime, $b_k \equiv 1 \pmod n$
($b_k \equiv 1 \pmod n$ is covered by ②)

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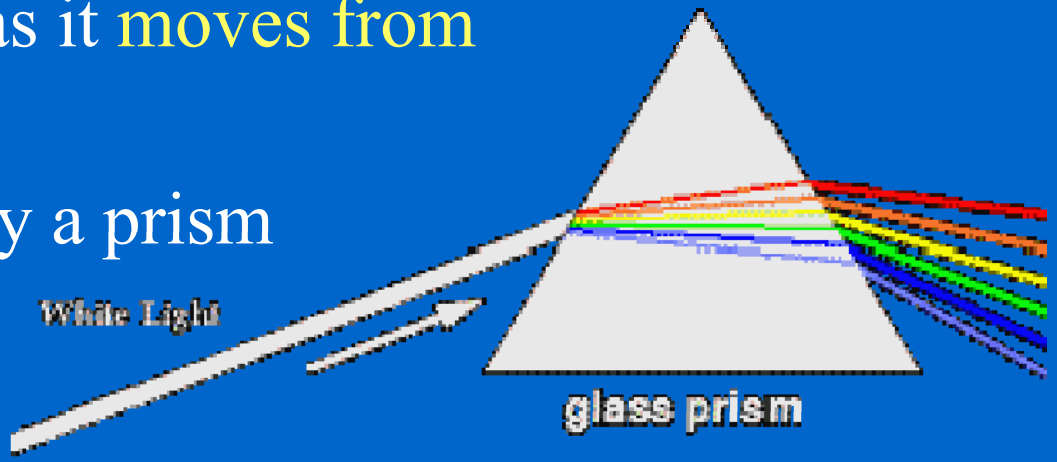

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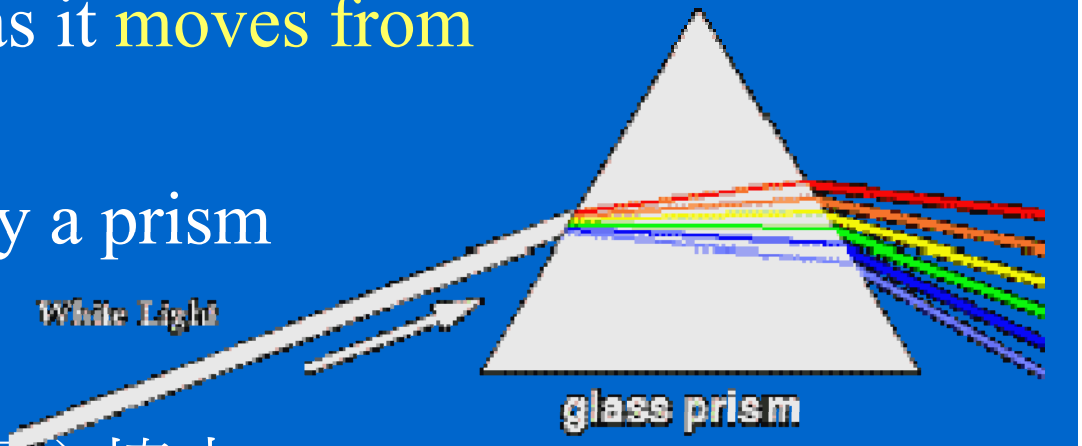

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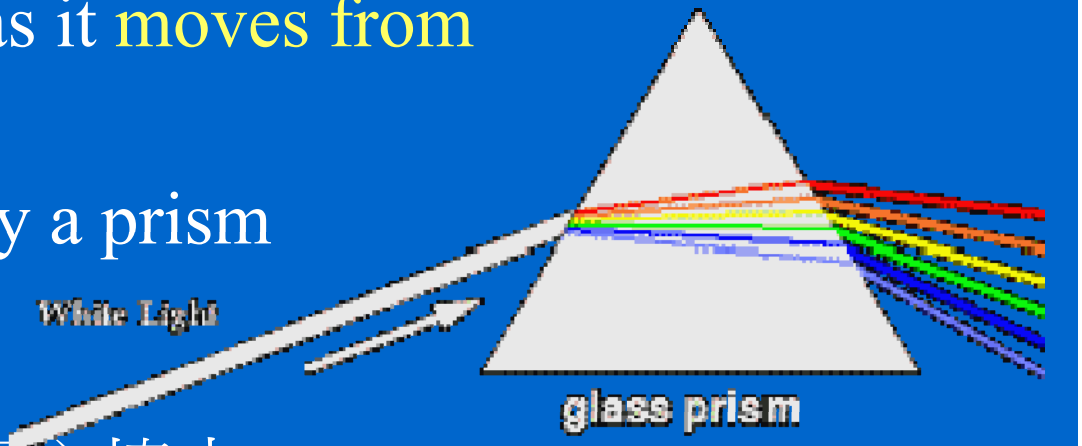

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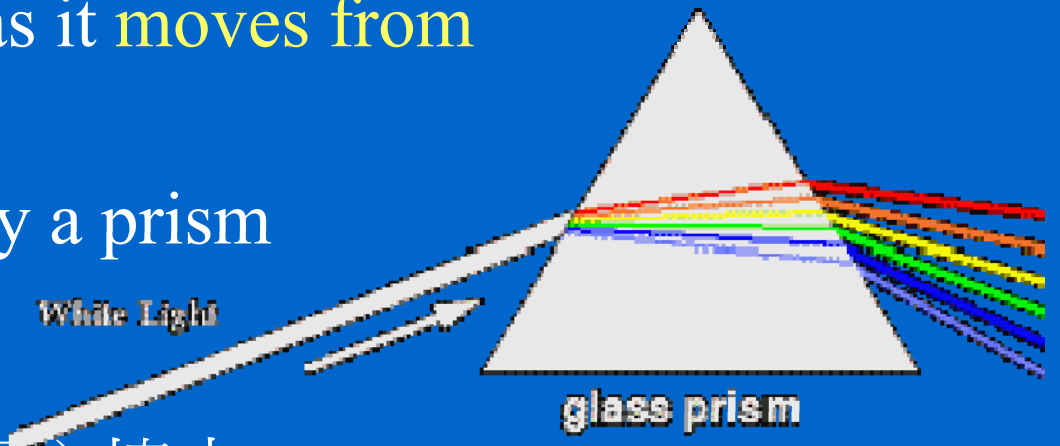

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	2^2	2^3	2^4	2^5	2^6	2^7	2^8
mod 11	4	8	**5**	**10**	**9**	**7**	**3**
mod 13	4	8	**3**	**6**	**12**	**11**	**9**

When/How does Basic Factoring Principle work in M-R test?

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  - catching the moment that $b_0, b_1, \ldots$ behave differently while taking square in (mod p) component and (mod q) component

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A Carmichael number: pass the Fermat test for all bases

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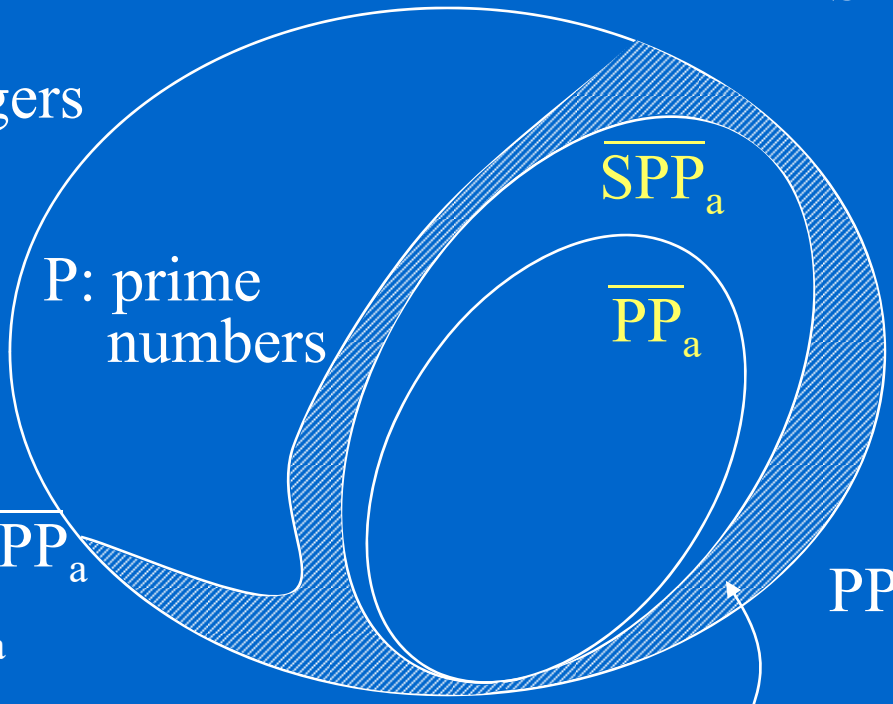
- Up to 10^{10}, there are 455052511 primes, there are 14884 pseudo prime numbers for the base 2, and 3291 strong pseudo prime numbers for the base 2

Fermat and Miller-Rabin Test

- Both of these two tests are for identifying subsets of composite numbers



SPP_a: strong pseudo prime numbers for base a, the set of composite n where M-T test says 'probably prime'

C: composite numbers

$I = P \cup C$

$C = SPP_a \cup \overline{SPP_a}$
$= PP_a \cup \overline{PP_a}$

$SPP_a \subset PP_a$

$\overline{PP_a} \subset \overline{SPP_a} \subset C$

PP_a: pseudo prime numbers for base a, the set of composite n where $a^{n-1} \equiv 1 \pmod{n}$

: mysterious part
not prime, but cannot be identified as composite

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- However, there are other kind of witness that n is composite, e.g., "$2^{n-1} \pmod n$ does not equal to 1" is also a witness that n is composite.
- A composite number will be factored out by the M-R test only if it is a pseudo prime but it is not a strong pseudo prime number.

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  - Does this meant that RSA was broken?
- Randomized algorithms like Rabin-Miller are far more efficient than the IIT algorithm, so we'll keep using those

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- maple('a:=nextprime(18973453578 9)')

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- By-product of Miller-Rabin primality test:
  - if n is a pseudoprime and not a strong pseudoprime, Miller-Rabin test can factor it. about 10^{-6} chance

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  This method works only when k_1 does not equal k_2.

Universal Exponent Factorization

- E.g.

  n=21146370779620657 1; e=9007; d=116402471153538991

  r=e*d-1=1048437057679925691936; powermod(2,r,n)=1

  let $r=2^5*r1$; r1=32763658052497677873

  powermod(2,r1,n)=187568564780117371 1

  powermod(2,2*r1,n)=113493629663725812 1

  powermod(2,4*r1,n)=1 => gcd(2*r1-1,n)=885320963 is a factor

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- Exponent factorization even if *r* is valid for one *a*, you can still try the above procedure

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If $n=p \cdot q$, $p-1$ and $q-1$ both have small factors that are less than B, then $\gcd(b-1,n)=n$, (useless) however, $b \equiv a^{B!} \equiv 1 \pmod n$ and we can use the Universal exponent method

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- Best records: p-1: 34 digits (113 bits), ECM: 47 digits (143 bits)

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  - the other factor is 3837523/1093 = 3511

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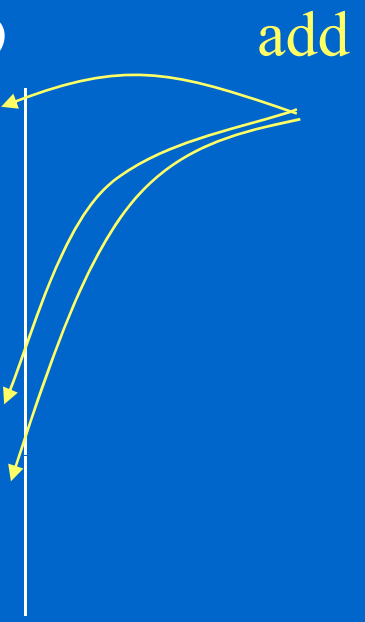

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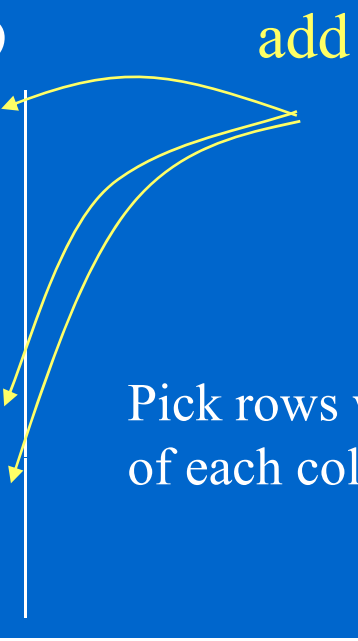

Pick rows where sums of each column are even

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Quadratic Sieve (3/4)

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- Looking for those $x^2 \equiv y^2$ but $x \neq \pm y$

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  - given RSA modulus n, public exponent e, ciphertext c

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Next challenge
RSA-640

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  - If we have the private exponent d, we can factor the modulus.

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  - Note: There are four roots to the equation $b^2 \equiv 1 \pmod{n}$, ± 1 are two of them, all satisfy $(b+1)(b-1) = k \cdot n = k \cdot p \cdot q$, since $0<b-1<b+1<n$, we have either ($p \mid b-1$ and $q \mid b+1$) or ($q \mid b-1$ and $p \mid b+1$), therefore, one of the factor can be found by $\gcd(b-1,n)$ and the other by $n/\gcd(b-1,n)$ or $\gcd(b+1,n)$

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  p = gcd(b-1,n) = gcd(5371,26069) = 131, q = n/p = 199

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- The above result says that “if you can recover a pair of RSA keys, you can factoring the corresponding $n=p \cdot q$” i.e. “once a private key d is compromised, you need to choose a new pair of (n, e) instead of changing e only”

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- The above result suggests that a scheme using (n, e_1), (n, e_2), … (n, e_k) with a common n for each k participants without giving each one the value of p, q is insecure. You should not use the same n as some others even though you are not explicitly told the value of p and q.

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- The above result also suggests that if you can recover arbitrary RSA key pair, you can solve the problem of factoring n. Whenever you get an **n**, you can form an RSA system with some ***e*** (assuming gcd(e, ϕ(n))=1), then use your method to solve the private exponent ***d*** without knowing p and q, after that you can factor n.

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different things

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- Add randomness through padding

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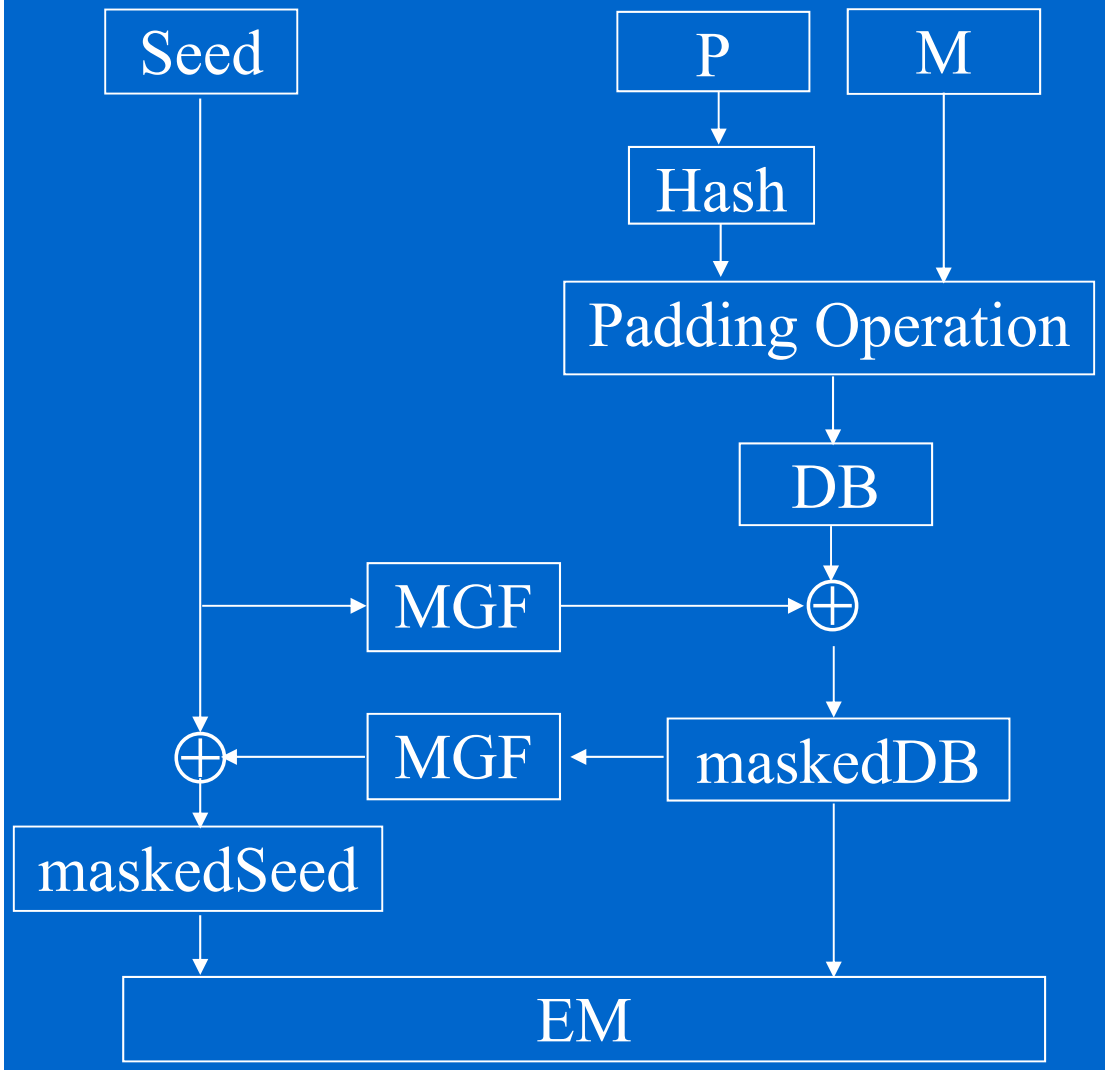
- c is now random corresponding to a fixed m, however, this only adds difficulties to the compilation of ciphertexts (a factor of 2^{64} times if PS is 8 bytes)

PKCS #1 v2 padding - OAEP



M: message (emLen-1-2hLen bytes)
P: encoding parameters,
an octet string
MGF: mask generation function
Hash: selected hash function
(hLen is the output bytes)
DB=Hash(P)||PS||01||M
PS is length emLen-
||M||-2hLen-1 null bytes
Seed: hLen random bytes
dbMask: MGF(seed, emLen-hLen)
maskedDB = DB ⊕ dbMask
seedMask:
MFG(maskedDB, hLen)
maskedSeed = seed ⊕ seedMask

EM: encoded message (emLen bytes)
EM = maskedSeed||makedDB

PKCS #1 v2 padding - OAEP

- Optimal Asymmetric Encryption (OAE)
  - M. Bellare, "Optimal Asymmetric Encryption - How to Encrypt with RSA," Eurocrypt'94
- Optimal Padding in the sense that
  - RSA-OAEP is semantically secure against adaptive chosen ciphertext attackers in the random oracle model
  - the message size in a k-bit RSA block is as large as possible (make the most advantage of the bandwidth)
- Following by more efficient padding schemes:
  - OAEP$^+$, SAEP$^+$, REACT

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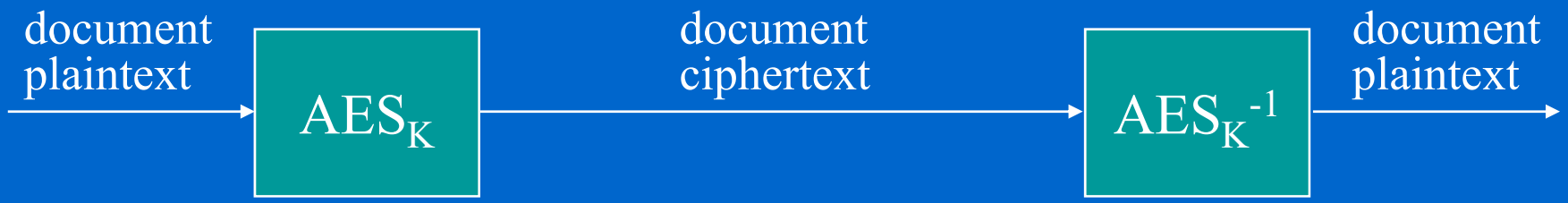

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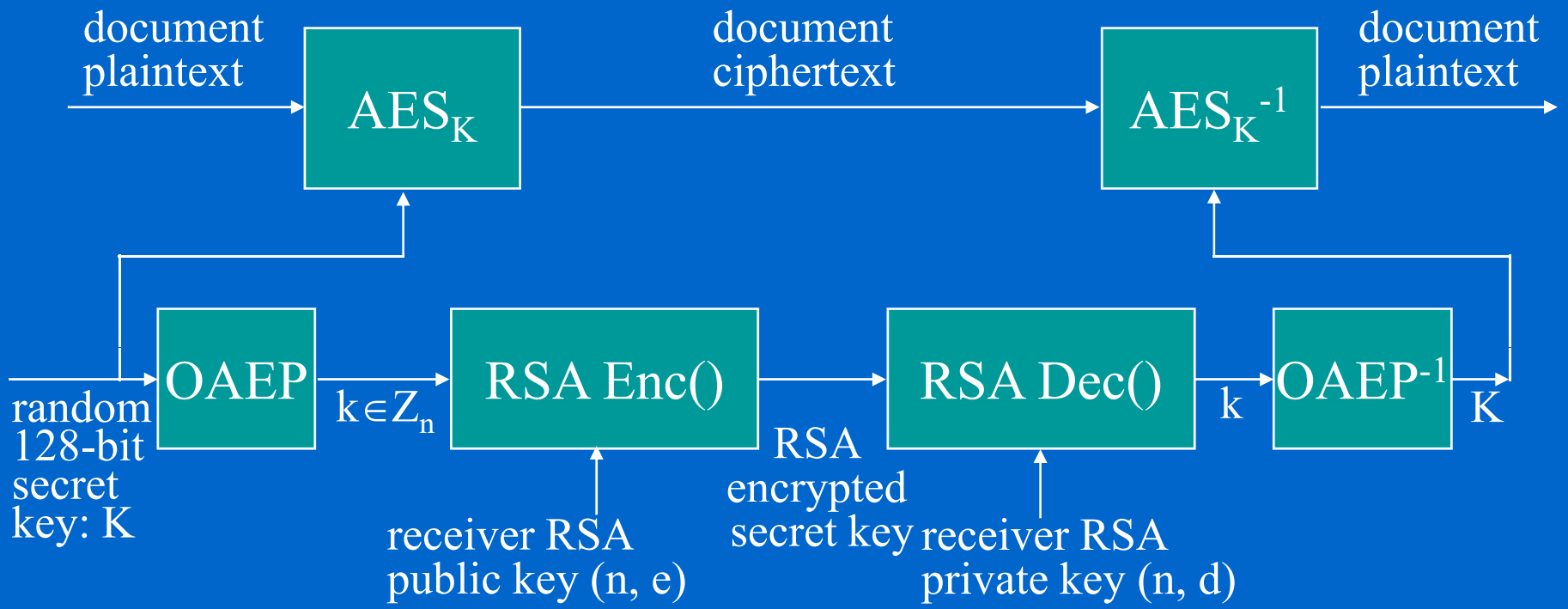

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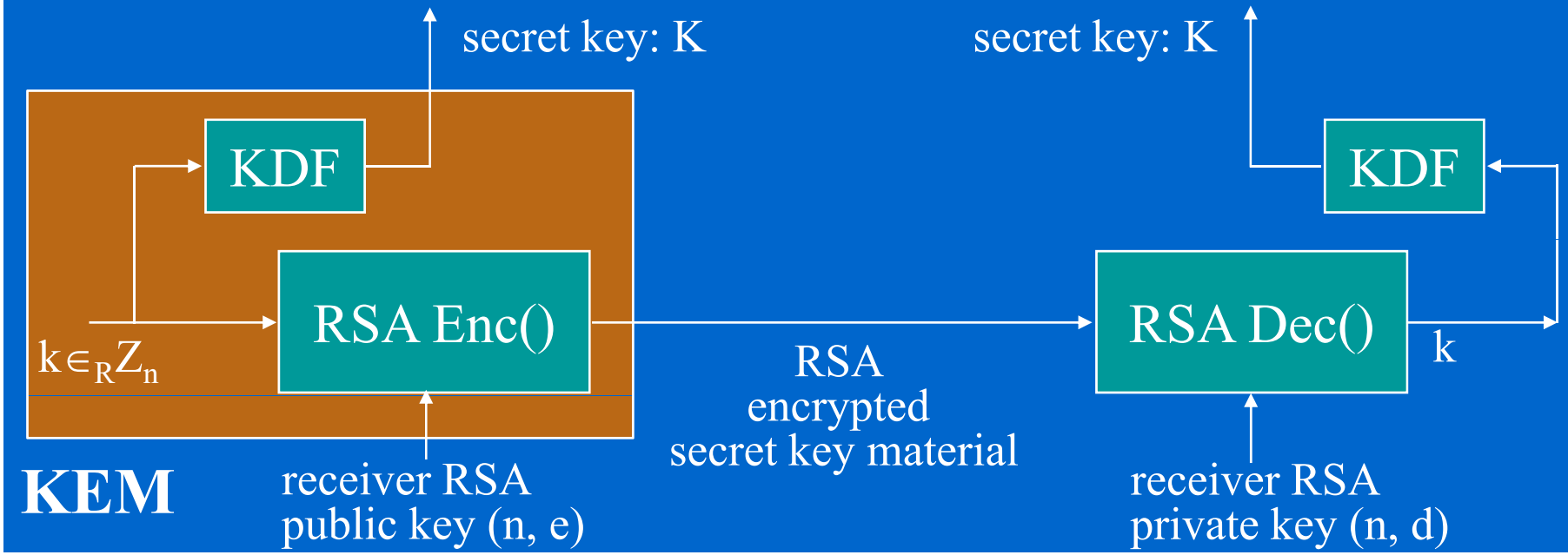

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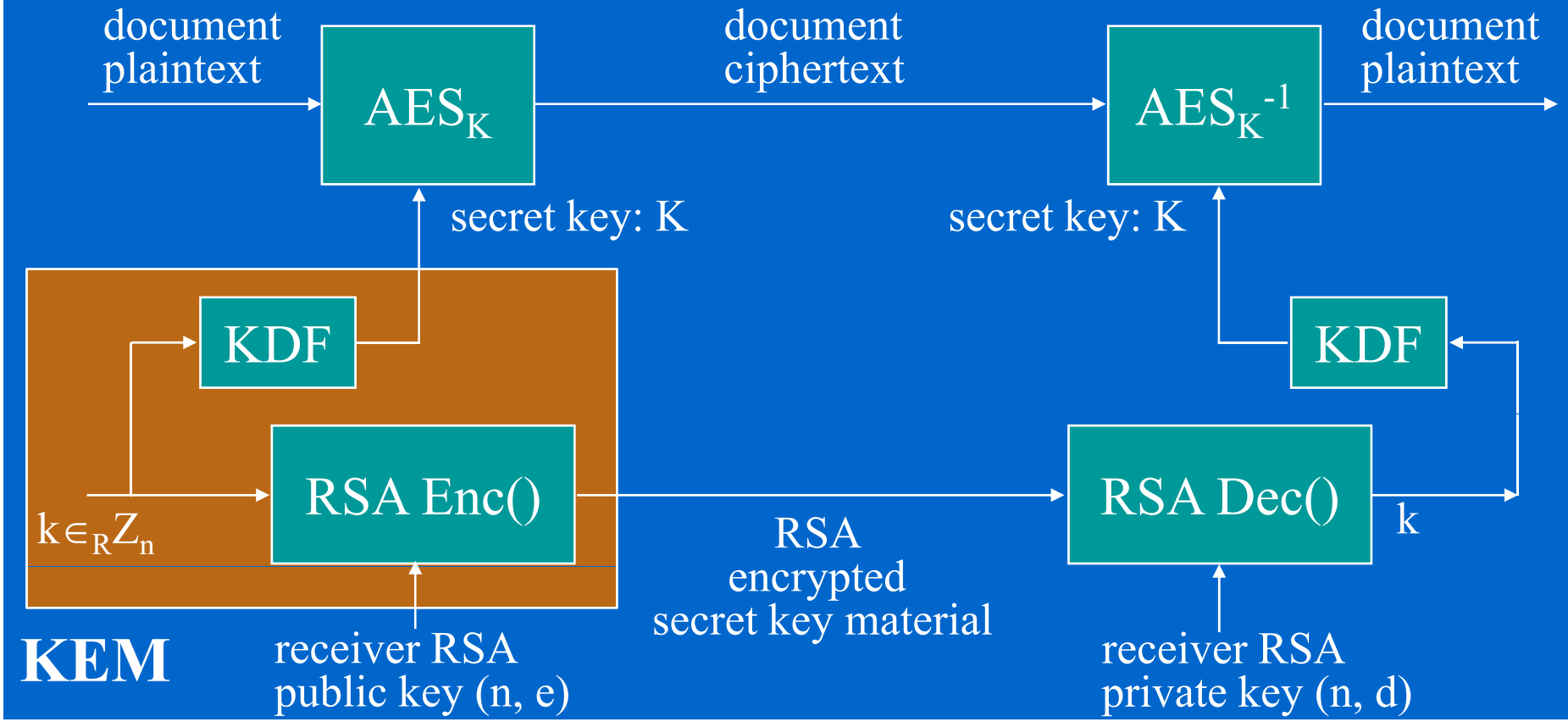

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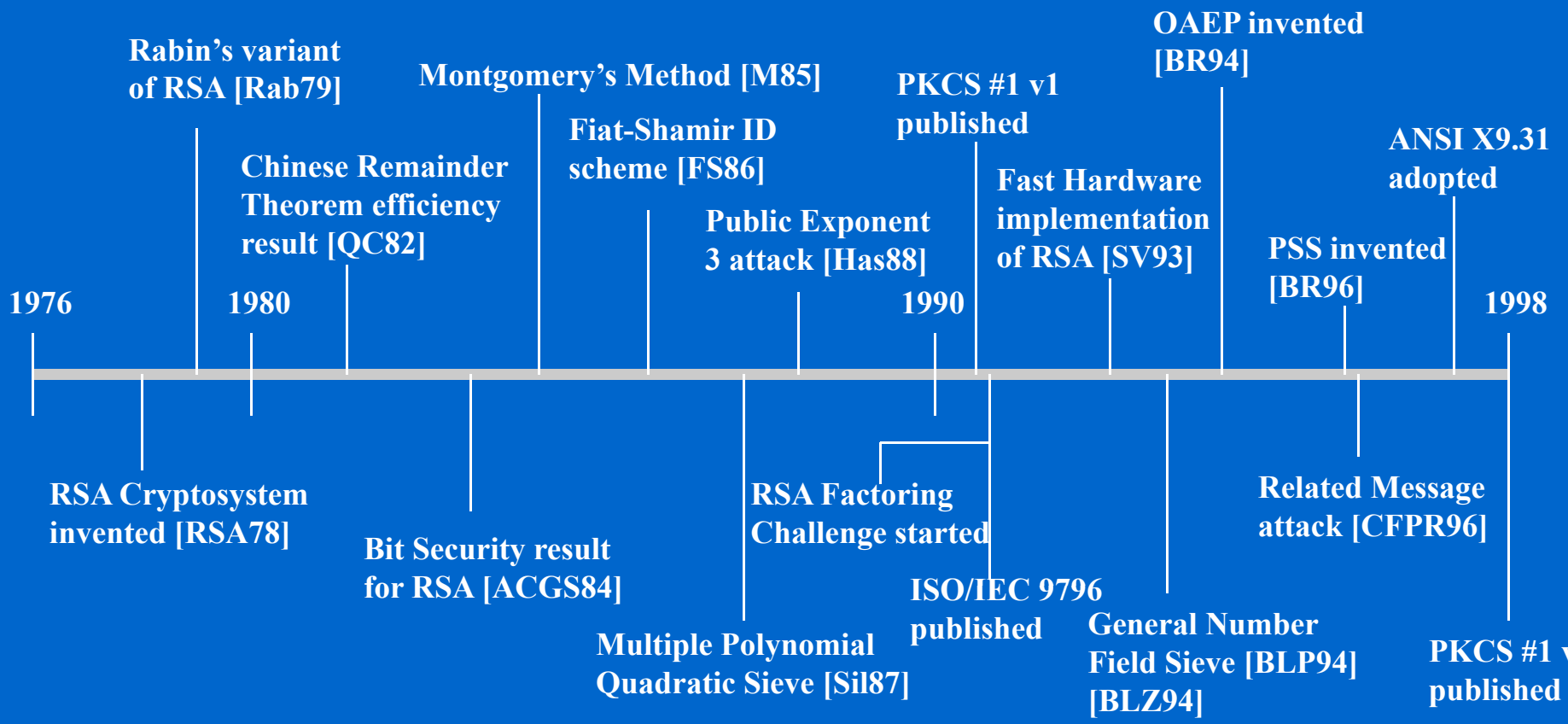
- advantages: lower computational cost for the decryption (and signature) primitives if CRT is used (also see **6.8.14**)

Factoring & RSA Timeline



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- Paillier Cryptosystem (High order composite residue based)
- Goldwasser-Micali Cryptosystem (QR based)
  - very low efficiency