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- ex. $\phi(10)=(2-1)\cdot(5-1)=4$ $\phi(120)=120(1-1/2)(1-1/3)(1-1/5)=32$

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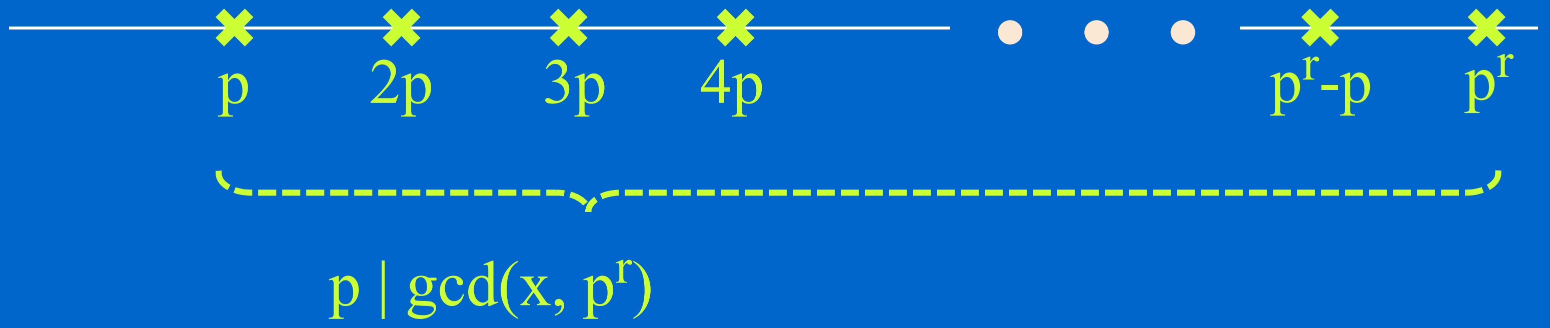

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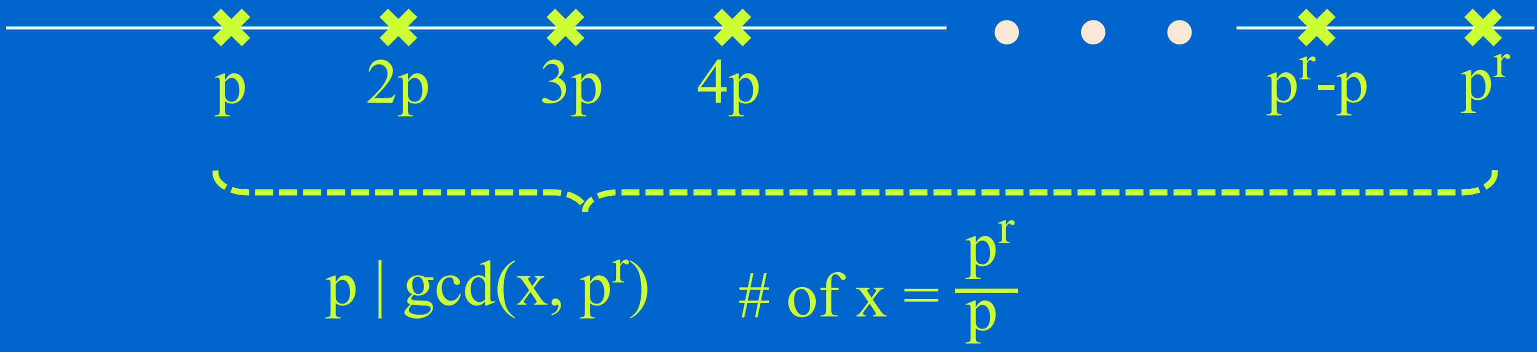

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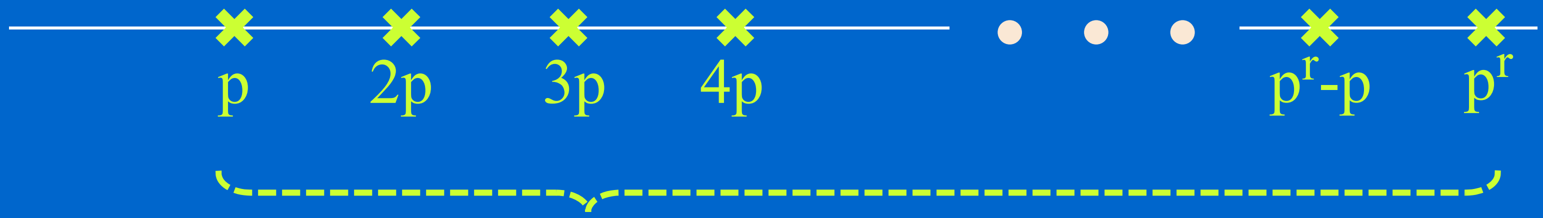

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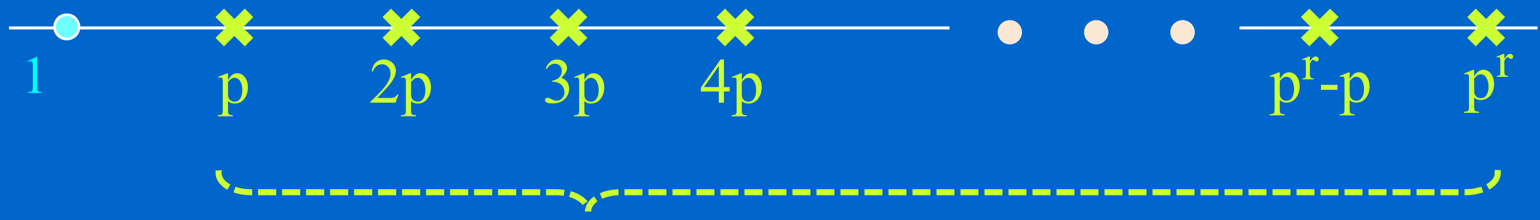

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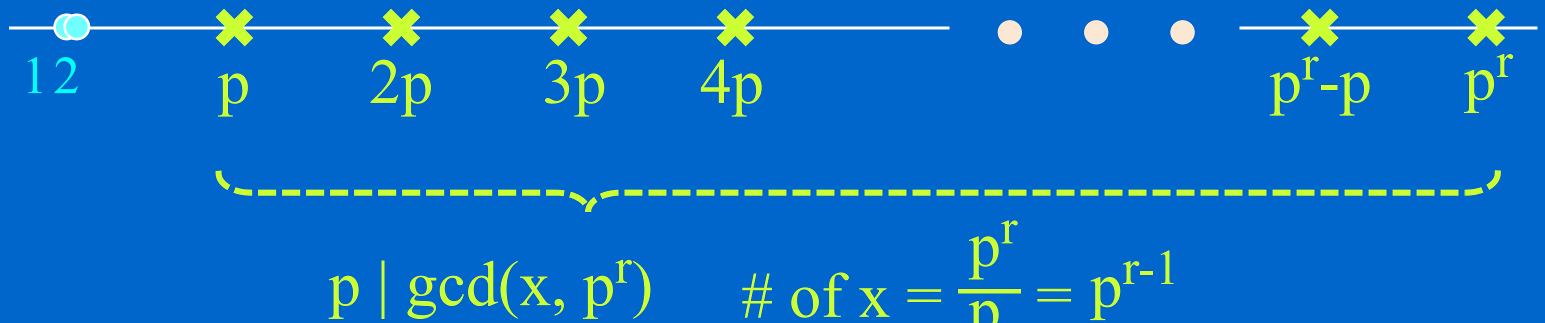

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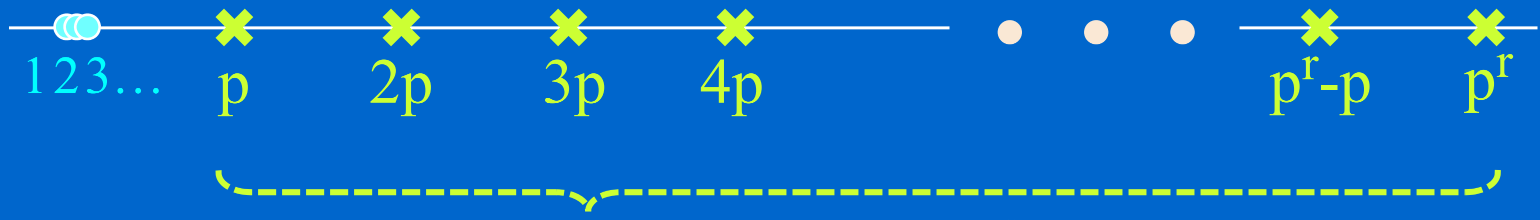

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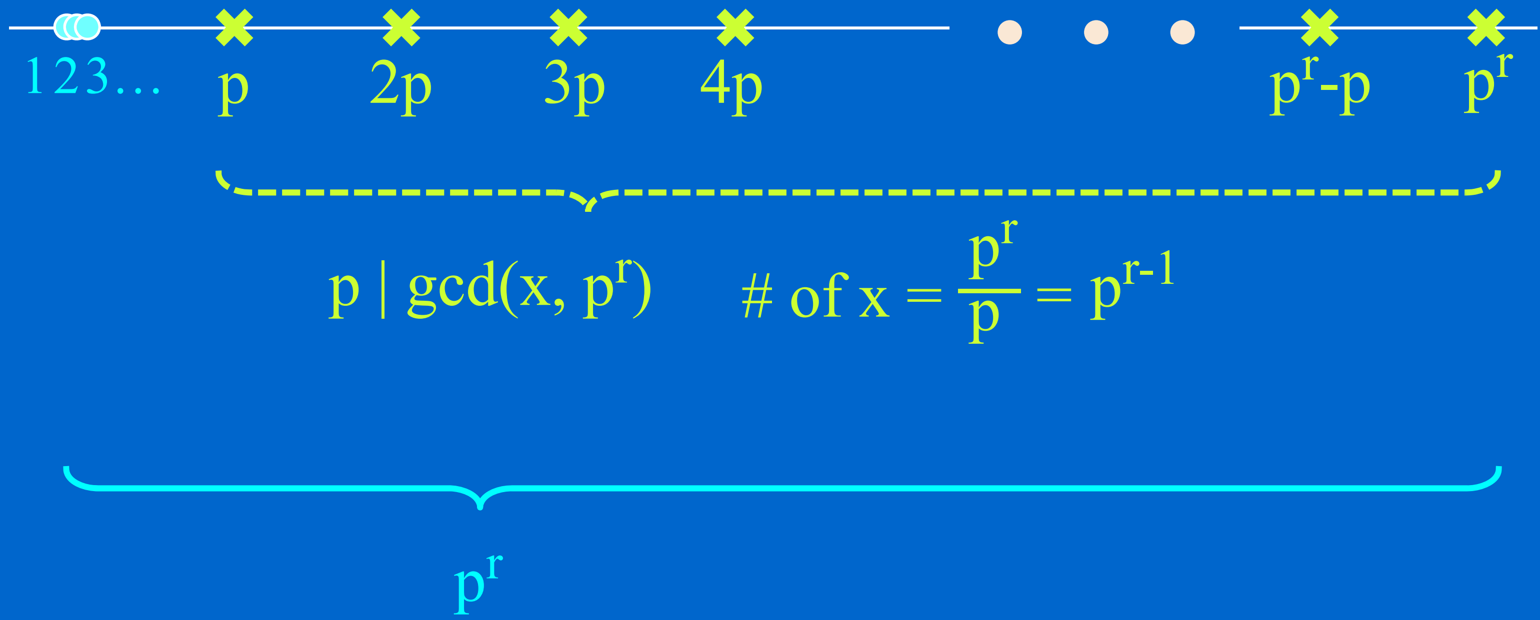

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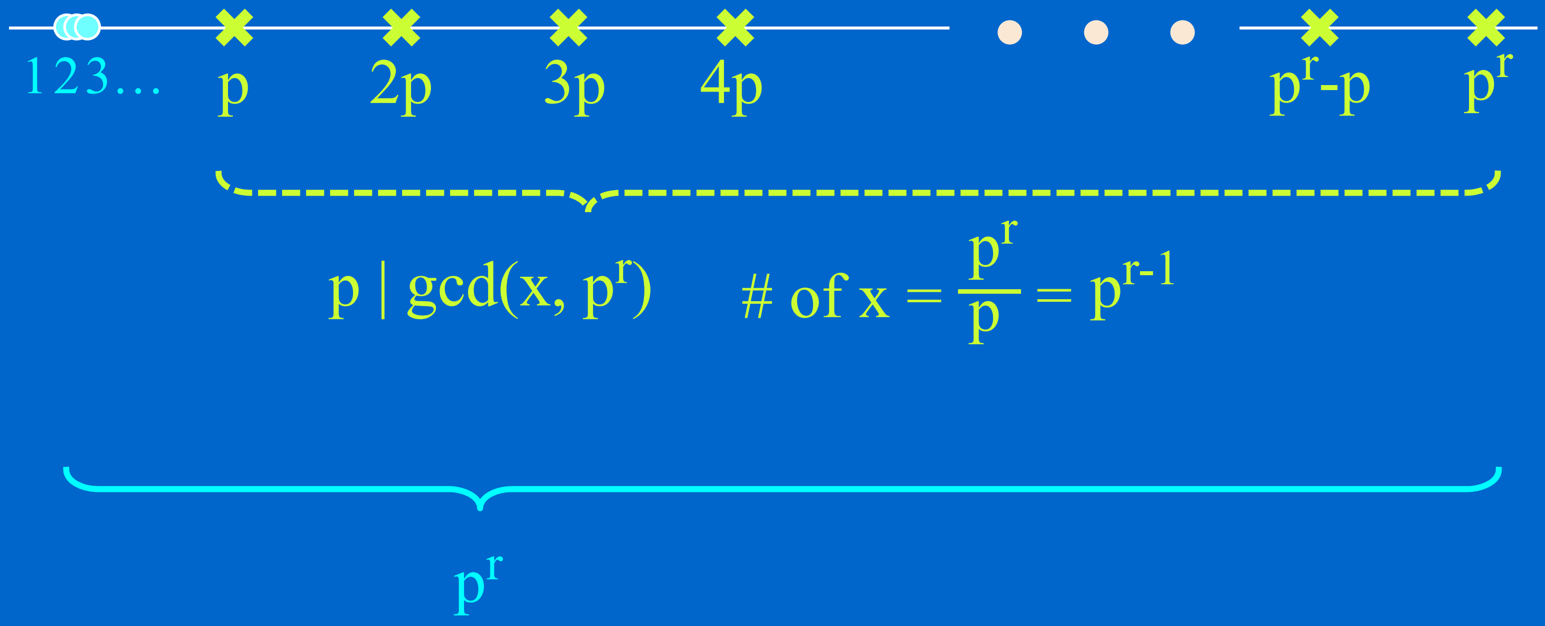

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Through CRT, each one of $\phi(n)\phi(m)$ pairs, i.e. (x_n, x_m), uniquely maps to an x in Z_{nm} which is relatively prime to n m

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&= (p_1^{c_1} - p_1^{c_1-1}) \cdot \phi(p_2^{c_2} \cdots p_k^{c_k}) \\
&= p_1^{c_1}(1-1/p_1) \cdot \phi(p_2^{c_2} \cdots p_k^{c_k}) \\
&= p_1^{c_1}(1-1/p_1) \cdot p_2^{c_2}(1-1/p_2) \cdot \phi(p_3^{c_3} \cdots p_k^{c_k})
\end{aligned}$$

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$$= \ldots$$

$\phi(n) = n \prod_{\forall p|n} (1-1/p)$

from **Unique Prime Factorization Theorem**: $n = p_1^{c_1} p_2^{c_2} \cdots p_k^{c_k}$

from **Euler totion function's multiplicative property**:

$$\phi(n\,m) = \phi(n) \cdot \phi(m)$$

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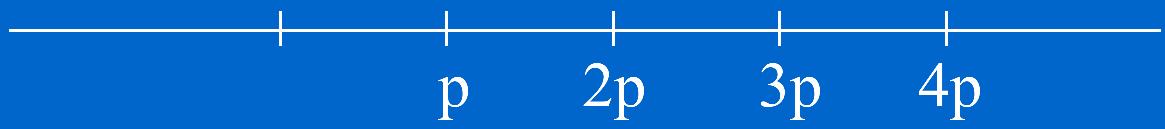

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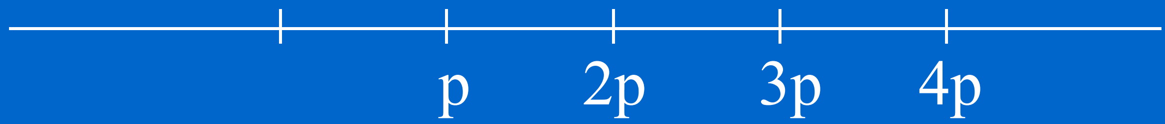

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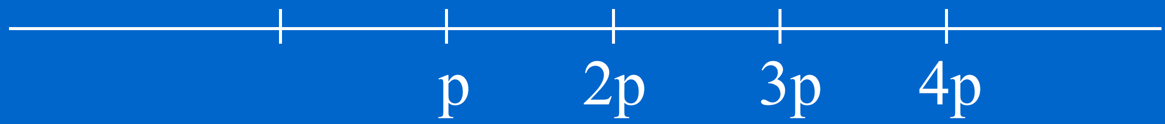

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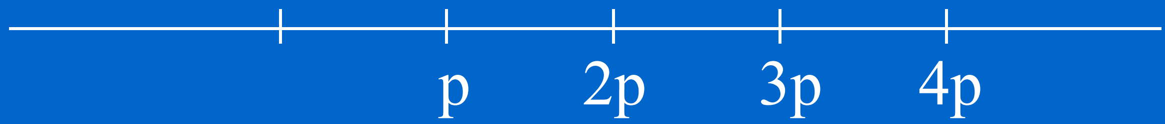

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- Equalities:

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  - Ex. n=4, $\zeta(4) = \pi^4/90 \approx 0.92$

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Special case: $\phi(p-1)$ x's in Z_p^* with $\text{ord}_p(x)=p-1$,
i.e. $\phi(p-1)$ generators in Z_p^*

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- ✧ Those a^ℓ with $\gcd(\ell, k) = d > 1$ have order at most ***k/d***
- ✧ Only the order of those a^ℓ with $\gcd(\ell, k) = 1$ might be k

e.g. $p = 13$

$\{2, 4, 8, 3, 6, 12, 11, 9, 5, 10, 7, 1\}$

2 is a generator in $Z_{13}^* = \{2^1, 2^2, 2^3, 2^4, 2^5, 2^6, 2^7, 2^8, 2^9, 2^{10}, 2^{11}, 2^{12}\}$

k=12, {2, ~~4~~, ~~8~~, ~~3~~, 6, ~~12~~, 11, ~~9~~, ~~5~~, ~~10~~, 7, ~~1~~}

k=6, {4, 3, 12, 9, 10, 1}

of ord-*k* elements in Z_p^*

Lemma. There are at most $\phi(k)$ ord-*k* elements in Z_p^*, $k \mid p$-1

pf.
- ✧ If $\boldsymbol{a}$ is an ord-*k* element in Z_p^*, then <$\boldsymbol{a}$> = {$\boldsymbol{a}^1$, $\boldsymbol{a}^2$, …, $\boldsymbol{a}^{k\text{-}1}$, $\boldsymbol{a}^k$=1} is a subgroup G, |G|=*k* spanned by $\boldsymbol{a}$.
- ✧ Those a^ℓ with gcd(ℓ, k) = d > 1 have order at most ***k*/*d***
- ✧ Only the order of those a^ℓ with gcd(ℓ, k) = 1 might be k
- ✧ Hence, there are at most $\phi(k)$ order k elements

e.g. p = 13

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of ord-k elements in Z_p^*

Lemma. There are at most $\phi(k)$ ord-k elements in Z_p^*, $k \mid p$-1

pf.

- ✧ If $\boldsymbol{a}$ is an ord-k element in Z_p^*, then $<\boldsymbol{a}> = \{\boldsymbol{a}^1, \boldsymbol{a}^2, \ldots, \boldsymbol{a}^{k-1}, \boldsymbol{a}^k=1\}$ is a subgroup G, |G|=k spanned by $\boldsymbol{a}$.
- ✧ Those a^ℓ with $\gcd(\ell, k) = d > 1$ have order at most ***k/d***
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Lemma. There are at most $\phi(k)$ ord-k elements in Z_p^*, $k \mid p-1$

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- ✧ If $\boldsymbol{a}$ is an ord-k element in Z_p^*, then $<\boldsymbol{a}> = \{\boldsymbol{a}^1, \boldsymbol{a}^2, \ldots, \boldsymbol{a}^{k-1}, \boldsymbol{a}^k = 1\}$ is a subgroup G, $|G| = k$ spanned by $\boldsymbol{a}$.
- ✧ Those a^ℓ with $\gcd(\ell, k) = d > 1$ have order at most ***k/d***
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of ord-k elements in Z_p^*

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- ✧ Those a^ℓ with $\gcd(\ell, k) = d > 1$ have order at most $\boldsymbol{k/d}$
- ✧ Only the order of those a^ℓ with $\gcd(\ell, k) = 1$ might be k
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k=6, {4, ~~3~~, ~~12~~, ~~9~~, 10, ~~1~~}, $\phi(6)$

of ord-k elements in Z_p^*

Lemma. There are at most $\phi(k)$ ord-k elements in Z_p^*, $k \mid p$-1

pf.
- ✧ If $\boldsymbol{a}$ is an ord-k element in Z_p^*, then $<\boldsymbol{a}> = \{\boldsymbol{a}^1, \boldsymbol{a}^2, \ldots, \boldsymbol{a}^{k-1}, \boldsymbol{a}^k=1\}$ is a subgroup G, |G|=k spanned by $\boldsymbol{a}$.
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k=6, {4, ~~3~~, ~~12~~, ~~9~~, 10, ~~1~~}, $\phi(6)$

k=4, {8, 12, 5, 1}, $\phi(4)$

of ord-*k* elements in Z_p^*

Lemma. There are at most $\phi(k)$ ord-k elements in Z_p^*, $k \mid p\text{-}1$

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- ✧ If $\boldsymbol{a}$ is an ord-k element in Z_p^*, then $<\boldsymbol{a}> = \{\boldsymbol{a}^1, \boldsymbol{a}^2, \ldots, \boldsymbol{a}^{k-1}, \boldsymbol{a}^k = 1\}$ is a subgroup G, |G|=k spanned by $\boldsymbol{a}$.
- ✧ Those a^ℓ with $\gcd(\ell, k) = d > 1$ have order at most $\boldsymbol{k/d}$
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k=6, {4, ~~3~~, ~~12~~, ~~9~~, 10, ~~1~~}, $\phi(6)$

k=4, {8, ~~12~~, 5, 1}, $\phi(4)$

of ord-k elements in Z_p^*

Lemma. There are at most $\phi(k)$ ord-k elements in Z_p^*, $k \mid p$-1

pf.
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k=4, {8, ~~12~~, 5, ~~1~~}, $\phi(4)$

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k=4, {8, ~~12~~, 5, ~~1~~}, $\phi(4)$

k=3, {3, 9, 1}, $\phi(3)$

of ord-k elements in Z_p^*

<u>Lemma</u>. There are at most $\phi(k)$ ord-k elements in Z_p^*, $k \mid p-1$

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k=4, {8, ~~12~~, 5, ~~1~~}, $\phi(4)$

k=3, {3, 9, ~~1~~}, $\phi(3)$

of ord-k elements in Z_p^*

Lemma. There are at most $\phi(k)$ ord-k elements in Z_p^*, $k \mid p-1$

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k=4, {8, ~~12~~, 5, ~~1~~}, $\phi(4)$

k=3, {3, 9, ~~1~~}, $\phi(3)$

k=2, {12,1}, $\phi(2)$

of ord-k elements in Z_p^*

Lemma. There are at most $\phi(k)$ ord-k elements in Z_p^*, $k \mid p-1$

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k=6, {4, ~~3~~, ~~12~~, ~~9~~, 10, ~~1~~}, $\phi(6)$

k=4, {8, ~~12~~, 5, ~~1~~}, $\phi(4)$

k=3, {3, 9, ~~1~~}, $\phi(3)$

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k=6, {4, ~~3~~, ~~12~~, ~~9~~, 10, ~~1~~}, $\phi(6)$

k=4, {8, ~~12~~, 5, ~~1~~}, $\phi(4)$

k=3, {3, 9, ~~1~~}, $\phi(3)$

k=2, {12, ~~1~~}, $\phi(2)$

k=1, {1}, $\phi(1)$

$\sum_{k|p-1} \phi(k) = p-1$

Lemma. $\sum_{k|p-1} \phi(k) = p-1$

let $\phi(1)=1$

$\Sigma_{k|p-1}\ \phi(k) = p-1$

Lemma. $\Sigma_{k|p-1}\ \phi(k) = p-1$ let $\phi(1)=1$

pf. $p-1 = \Sigma_{k|p-1}$ (# a in Z_p^* s.t. $\gcd(a, p-1) = k$)

$\Sigma_{k|p-1}\, \phi(k) = p\text{-}1$

Lemma. $\Sigma_{k|p-1}\, \phi(k) = p-1$

let $\phi(1)=1$

pf. $p-1 = \Sigma_{k|p-1}$ (# a in Z_p^* s.t. $\gcd(a, p-1) = k$)

let $p=13$, $a \in Z_p^*$

$\gcd(a, p-1)=k$, i.e. $k \mid p-1$

$\Sigma_{k|p-1}\ \phi(k) = p\text{-}1$

Lemma. $\Sigma_{\ k|p-1}\ \phi(k) = p\text{-}1$

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pf. $p\text{-}1 = \Sigma_{\ k|p-1}$ (# a in Z_p^* s.t. gcd(a, p-1) = k)

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gcd(a, p-1)=k, i.e. $k \mid p\text{-}1$

k=1, {1,5,7,11}, $\phi(12/1)$

$\Sigma_{k|p\text{-}1}\ \phi(k) = p\text{-}1$

Lemma. $\Sigma_{k|p\text{-}1}\ \phi(k) = p\text{-}1$ let $\phi(1)=1$

pf. $p\text{-}1 = \Sigma_{k|p\text{-}1}$ (# a in Z_p^* s.t. gcd(a, p-1) = k)

let p=13, $a \in Z_p^*$

gcd(a, p-1)=k, i.e. $k \mid p$-1

k=1, {1,5,7,11}, ϕ(12/1)

k=2, {2,10}, ϕ(12/2)

$\Sigma_{k|p\text{-}1}\ \phi(k) = p\text{-}1$

Lemma. $\Sigma_{k|p\text{-}1}\ \phi(k) = p\text{-}1$

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gcd(a, p-1)=k, i.e. $k \mid p\text{-}1$

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$\Sigma_{k|p-1}\ \phi(k) = p\text{-}1$

Lemma. $\Sigma_{k|p-1}\ \phi(k) = p-1$

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let p=13, $a \in Z_p^*$

gcd(a, p-1)=k, i.e. $k \mid p-1$

k=1, {1,5,7,11}, ϕ(12/1)

k=2, {2,10}, ϕ(12/2)

k=3, {3,9}, ϕ(12/3)

k=4, {4,8}, ϕ(12/4)

$\Sigma_{k|p\text{-}1}\ \phi(k) = p\text{-}1$

Lemma. $\Sigma_{\,k|p\text{-}1}\ \phi(k) = p\text{-}1$ let $\phi(1)=1$

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$\gcd(a, p\text{-}1)=k$, i.e. $k \mid p\text{-}1$

k=1, {1,5,7,11}, $\phi(12/1)$

k=2, {2,10}, $\phi(12/2)$

k=3, {3,9}, $\phi(12/3)$

k=4, {4,8}, $\phi(12/4)$

k=6, {6}, $\phi(12/6)$

$\Sigma_{k|p\text{-}1}\ \phi(k) = p\text{-}1$

Lemma. $\Sigma_{k|p-1}\ \phi(k) = p-1$

let $\phi(1)=1$

pf. $p-1 = \Sigma_{k|p-1}$ (# a in Z_p^* s.t. $\gcd(a, p-1) = k$)

let $p=13$, $a \in Z_p^*$

$\gcd(a, p-1)=k$, i.e. $k \mid p-1$

$k=1$, $\{1,5,7,11\}$, $\phi(12/1)$

$k=2$, $\{2,10\}$, $\phi(12/2)$

$k=3$, $\{3,9\}$, $\phi(12/3)$

$k=4$, $\{4,8\}$, $\phi(12/4)$

$k=6$, $\{6\}$, $\phi(12/6)$

$k=12$, $\{12\}$, $\phi(12/12)$

$\Sigma_{k|p\text{-}1}\, \phi(k) = p\text{-}1$

Lemma. $\Sigma_{k|p\text{-}1}\, \phi(k) = p\text{-}1$ let $\phi(1)=1$

pf. $p\text{-}1 = \Sigma_{k|p\text{-}1}$ (# a in Z_p^* s.t. $\gcd(a, p\text{-}1) = k$)

a/k

$= \Sigma_{k|p\text{-}1}$ (# b in $\{1,\ldots,(p\text{-}1)/k\}$ s.t. $\gcd(b, (p\text{-}1)/k) = 1$)

let $p=13$, $a \in Z_p^*$

$\gcd(a, p\text{-}1)=k$, i.e. $k \mid p\text{-}1$

$k=1$, $\{1,5,7,11\}$, $\phi(12/1)$

$k=2$, $\{2,10\}$, $\phi(12/2)$

$k=3$, $\{3,9\}$, $\phi(12/3)$

$k=4$, $\{4,8\}$, $\phi(12/4)$

$k=6$, $\{6\}$, $\phi(12/6)$

$k=12$, $\{12\}$, $\phi(12/12)$

$\Sigma_{k|p-1}\ \phi(k) = p-1$

Lemma. $\Sigma_{k|p-1}\ \phi(k) = p-1$ let $\phi(1)=1$

pf. $p-1 = \Sigma_{k|p-1}$ (# a in Z_p^* s.t. $\gcd(a, p-1) = k$)

a/k

$= \Sigma_{k|p-1}$ (# b in $\{1,\ldots,(p-1)/k\}$ s.t. $\gcd(b, (p-1)/k) = 1$)

$= \Sigma_{k|p-1}\ \phi((p-1)/k)$

let $p=13$, $a \in Z_p^*$

$\gcd(a, p-1)=k$, i.e. $k \mid p-1$

$k=1$, $\{1,5,7,11\}$, $\phi(12/1)$

$k=2$, $\{2,10\}$, $\phi(12/2)$

$k=3$, $\{3,9\}$, $\phi(12/3)$

$k=4$, $\{4,8\}$, $\phi(12/4)$

$k=6$, $\{6\}$, $\phi(12/6)$

$k=12$, $\{12\}$, $\phi(12/12)$

$\Sigma_{k|p-1}\ \phi(k) = p-1$

Lemma. $\Sigma_{k|p-1}\ \phi(k) = p-1$ let $\phi(1)=1$

pf. $p-1 = \Sigma_{k|p-1}$ (# a in Z_p^* s.t. $\gcd(a, p-1) = k$)

a/k

$= \Sigma_{k|p-1}$ (# b in $\{1,\ldots,(p-1)/k\}$ s.t. $\gcd(b, (p-1)/k) = 1$)

$= \Sigma_{k|p-1}\ \phi((p-1)/k)$

let $p=13$, $a \in Z_p^*$

$\gcd(a, p-1)=k$, i.e. $k \mid p-1$

$k=1$, $\{1,5,7,11\}$, $\phi(12/1)$

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$k=12$, $\{12\}$, $\phi(12/12)$

$\phi(1\} + \phi(12) +$

$\phi(2) + \phi(6) +$

$\phi(3) + \phi(4)$

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Z_p^* is a *cyclic* group

Theorem: Z_p^* is a ***cyclic*** group for a prime number p

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Ex. p=13, p-1 = |{2,6,11,7}| + |{4,10}| + |{8,5}| + |{3,9}| + |{12}| + |{1}|

k=12, k=6, k=4, k=3, k=2, k=1

$Z^*_{p^s}$ is cyclic

- $Z^*_{p^s} = \{1, 2, \ldots, p-1,$
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pf. (by contradiction, for each i=s-2, s-3, …, 1)

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$(g^k)^p \equiv (1+\lambda p^{s-2})^p \equiv 1 \pmod{p^{s-1}}$, where $kp < p^{s-2}(p-1)$

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$(g^k)^p \equiv (1+\lambda p^{s-2})^p \equiv 1 \pmod{p^{s-1}}$, where $kp < p^{s-2}(p\text{-}1)$

i.e. g is not a generator in $Z^*_{p^{s-1}}$, contradiction with ③

$Z^*_{p^s}$ is cyclic (cont'd)

⑤ let $n=\text{ord}_{p^s}(g)$, Euler's Thm $g^{p^{s-1}(p-1)} \equiv 1 \pmod{p^s} \Rightarrow n \mid p^{s-1}(p-1)$

$Z^*_{p^s}$ is cyclic (cont'd)

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⑤,⑥ $\Rightarrow n = p^{s-2}(p-1)$ or $n = p^{s-1}(p-1)$

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⑦ $\text{ord}_{p^{s-2}}(g) \overset{④}{=} p^{s-3}(p-1) \Rightarrow \exists\, \lambda,\ g^{p^{s-3}(p-1)} = 1+\lambda p^{s-2}$

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⑦ $\mathrm{ord}_{p^{s-2}}(g) = p^{s-3}(p-1) \Rightarrow \exists\, \lambda,\ g^{p^{s-3}(p-1)} = 1+\lambda p^{s-2}$

$\mathrm{ord}_{p^{s-1}}(g) = p^{s-2}(p-1) \Rightarrow g^{p^{s-3}(p-1)} \neq 1 \pmod{p^{s-1}}$

$Z^*_{p^s}$ is cyclic (cont'd)

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$\Rightarrow p \nmid \lambda$

$Z^*_{p^s}$ is cyclic (cont'd)

⑤ let $n = ord_{p^s}(g)$, Euler's Thm $g^{p^{s-1}(p-1)} \equiv 1 \pmod{p^s} \Rightarrow n \mid p^{s-1}(p-1)$

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$$(g^{p^{s-3}(p-1)})^p \equiv (1+\lambda p^{s-2})^p \equiv 1 + p\lambda p^{s-2} + C_2^p \lambda^2 (p^{s-2})^2 + \ldots$$

$$\equiv 1+\lambda p^{s-1} \pmod{p^s}$$

$Z_{p^s}^*$ is cyclic (cont'd)

⑤ let $n=\mathrm{ord}_{p^s}(g)$, Euler's Thm $g^{p^{s-1}(p-1)} \equiv 1 \pmod{p^s} \Rightarrow n \mid p^{s-1}(p-1)$

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⑤,⑥ $\Rightarrow n = p^{s-2}(p-1)$ or $n = p^{s-1}(p-1)$

④

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Quadratic Residue modulo p^s

- For each $x \in \mathbf{Z}^*_{p^s}$, p^s-$x \neq x \pmod{p^s}$ (since if x is odd, p^s-x is even), it's clear that x and p^s-x are both square roots of a certain $y \in \mathbf{Z}^*_{p^s}$

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- let b be a quadratic residue mod p and mod p^s

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Tonelli's 1891 note

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Square root mod n

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  - there are 2^k square roots