Prime Numbers

密碼學與應用

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Prime Numbers

✧ **Prime number**: an integer p>1 that is divisible only by 1 and itself, ex. 2, 3,5, 7, 11, 13, 17…

✧ **Composite number**: an integer n>1 that is not prime

✧ **Fact**: there are infinitely many prime numbers. (by Euclid)

pf:
- ✡ on the contrary, assume a_n is the largest prime number
- ✡ let the finite set of prime numbers be $\{a_0, a_1, a_2, \ldots\ a_n\}$
- ✡ the number $b = a_0 * a_1 * a_2 * \ldots * a_n + 1$ is not divisible by any a_i
  i.e. b does not have prime factors $\leq a_n$

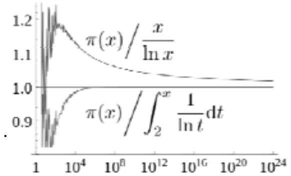
2 cases:
- ➢ if b has a prime factor d, $b > d > a_n$, then "d is a prime number that is larger than a_n" … contradiction
- ➢ if b does not have any prime factor less than b, then "b is a prime number that is larger than a_n" … contradiction

Prime Number Theorem

✧ **Prime Number Theorem**:
- ✶ Let $\pi(x)$ be the number of primes less than x
- ✶ Then
$$\pi(x) \approx \frac{x}{\ln x}$$
in the sense that the ratio $\pi(x) / (x/\ln x) \to 1$ as $x \to \infty$
- ✶ Also, $\pi(x) \geq \frac{x}{\ln x}$ and for $x \geq 17$, $\pi(x) \leq 1.10555 \frac{x}{\ln x}$

✧ Ex: number of 100-digit primes

$$\pi(10^{100}) - \pi(10^{99}) \approx \frac{10^{100}}{\ln 10^{100}} -$$



Factors

✧ Every composite number can be expressible as a product a·b of integers with 1 < a, b< n

✧ Every positive integer has a unique representation as a product of prime numbers raised to different powers.
- ✡ Ex. $504 = 2^3 \cdot 3^2 \cdot 7$, $1125 = 3^2 \cdot 5^3$

Factors

- Lemma: p is a prime number and $p \mid a\cdot b \Longrightarrow p \mid a$ or $p \mid b$, more generally, p is a prime number and $p \mid a\cdot b\cdot\ldots\cdot z$ $\Longrightarrow$ p must divide one of a, b, …, z
  - proof:
    - case 1: $p \mid a$
    - case 2: $p \nmid a$,
      - $p \nmid a$ and p is a prime number $\Rightarrow \gcd(p, a) = 1 \Rightarrow 1 = a\,x + p\,y$
      - multiply both side by b, $b = \underline{b\,a}\,x + b\,\underline{p}\,y$
      - $p \mid a\,b \Rightarrow p \mid b$
    - In general: if $p \mid a$ then we are done, if $p \nmid a$ then $p \mid bc\ldots z$, continuing this way, we eventually find that p divides one of the factors of the product

Unique Prime Factorization Theorem

- Theorem: Every positive integer is a product of primes. This factorization into primes is unique, up to reordering of the factors.
  - Proof: product of primes
    - assume there exist positive integers that are not product of primes
    - let n be the smallest such integer
    - since n can not be 1 or a prime, n must be composite, i.e. $n = a\cdot b$
    - since n is the smallest, both a and b must be products of primes.
    - $n = a\cdot b$ must also be a product of primes, contradiction
  - Proof: uniqueness of factorization
    - assume $n = r_1^{c_1} r_2^{c_2}\cdots r_k^{c_k}\, p_1^{a_1} p_2^{a_2}\cdots p_s^{a_s} = r_1^{c_1} r_2^{c_2}\cdots r_k^{c_k}\, q_1^{b_1} q_2^{b_2}\cdots q_t^{b_t}$ where p_i, q_j are all distinct primes.
    - let $m = n / (r_1^{c_1} r_2^{c_2}\cdots r_k^{c_k})$
    - consider p_1 for example, since p_1 divide $m = q_1 q_1 .. q_1 q_2 \ldots q_t$, p_1 must divide one of the factors q_j, contradict the fact that "p_i, q_j are distinct primes"

> • Empty product equals 1.
> • Prime is a one factor product.

("Fair-MAH")

Fermat's Little Theorem

- If p is a prime, $p \nmid a$ then $a^{p-1} \equiv 1 \pmod p$

Proof:
- let $S = \{1, 2, 3, \ldots, p-1\}$ (Z_p^*), define $\psi(x) \equiv a \cdot x \pmod p$ be a mapping $\psi: S \to Z$
- $\forall x \in S, \psi(x) \neq 0 \pmod p \Rightarrow \forall x \in S, \psi(x) \in S$, i.e. $\psi: S \to S$
  - if $\psi(x) \equiv a \cdot x \equiv 0 \pmod p \Rightarrow x \equiv 0 \pmod p$ since $\gcd(a, p) = 1$
- $\forall x, y \in S$, if $x \neq y$ then $\psi(x) \neq \psi(y)$
  - if $\psi(x) \equiv \psi(y) \Rightarrow a \cdot x \equiv a \cdot y \Rightarrow x \equiv y$ since $\gcd(a, p) = 1$
- from the above two observations, $\psi(1), \psi(2), \ldots \psi(p-1)$ are distinct elements of S
- $1\cdot 2 \cdot\ldots\cdot(p-1) \equiv \psi(1)\cdot\psi(2)\cdot\ldots\cdot\psi(p-1) \equiv (a\cdot 1)\cdot(a\cdot 2)\cdot\ldots\cdot(a\cdot(p-1))$
  $\equiv a^{p-1}\,(1\cdot 2\cdot\ldots\cdot(p-1)) \pmod p$
- since $\gcd(j, p) = 1$ for $j \in S$, we can divide both side by 1, 2, 3, … p-1, and obtain $a^{p-1} \equiv 1 \pmod p$

Fermat's Little Theorem

- Ex: $2^{10} = 1024 \equiv 1 \pmod{11}$
  $2^{53} = (2^{10})^5 2^3 \equiv 1^5 2^3 \equiv 8 \pmod{11}$
  i.e. $2^{53} \equiv 2^{53 \bmod 10} \equiv 2^3 \equiv 8 \pmod{11}$

- if n is prime, then $2^{n-1} \equiv 1 \pmod n$
  i.e. if $2^{n-1} \neq 1 \pmod n$ then n is not prime $\leftarrow$(∗)
  usually, if $2^{n-1} \equiv 1 \pmod n$, then n is prime
  - exceptions: $2^{561-1} \equiv 1 \pmod{561}$ although $561 = 3\cdot 11\cdot 17$
    $2^{1729-1} \equiv 1 \pmod{1729}$ although $1729 = 7\cdot 13\cdot 19$
  - (∗) is a quick test for eliminating composite number

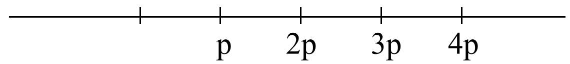
Euler's Totient Function $\phi(n)$

- $\phi(n)$: the number of integers $1 \le a < n$ s.t. $\gcd(a,n)=1$
  ex. $n=10$, $\phi(n)=4$ the set is $Z_{10}^* = \{1,3,7,9\}$
- properties of $\phi(\bullet)$
  - $\phi(p) = p-1$, if p is prime
  - $\phi(p^r) = p^r - p^{r-1} = p^r \cdot (1-1/p)$, if p is prime
  - $\phi(n\cdot m) = \phi(n) \cdot \phi(m)$ if $\gcd(n,m)=1$ *multiplicative property*
  - $\phi(n\cdot m) = \phi((d_1/d_2/d_3)^2)\cdot\phi(d_2^3)\cdot\phi(d_3^3)\cdot\phi(n/d_1/d_2)\cdot\phi(m/d_1/d_3)$
    if $\gcd(n,m)=d_1$, $\gcd(n/d_1,d_1)=d_2$, $\gcd(m/d_1,d_1)=d_3$
  - $\phi(n) = n \prod_{\forall p|n} (1-1/p)$

ex. $\phi(10)=(2-1)\cdot(5-1)=4$ $\phi(120)=120(1-1/2)(1-1/3)(1-1/5)=32$

How large is $\phi(n)$?

- $\phi(n) \approx n \cdot 6/\pi^2$ as n goes large
- Probability that a random number r is multiples of a prime number p? $1/p$ *think of 2 (even numbers), 3, 5, ...* **r** must be of the form **kp**



- Probability that two independent random numbers r_1 and r_2 both have a given prime number p as a factor? $1/p^2$
- The probability that they do not have p as a common factor is thus $1 - 1/p^2$
- The probability that two numbers r_1 and r_2 have no common prime factor? $P = (1-1/2^2)(1-1/3^2)(1-1/5^2)(1-1/7^2)\ldots$

Pr{ r_1 and r_2 relatively prime }

- Equalities: $\frac{1}{1-x} = 1+x+x^2+x^3+\ldots$

  $1 + 1/2^2 + 1/3^2 + 1/4^2 + 1/5^2 + 1/6^2 + \ldots = \pi^2/6$
- $P = (1-1/2^2)(1-1/3^2)(1-1/5^2)(1-1/7^2) \cdot \ldots$
  $= ((1+1/2^2+1/2^4+\ldots)(1+1/3^2+1/3^4+\ldots) \cdot \ldots)^{-1}$
  $= (1+1/2^2+1/3^2+1/4^2+1/5^2+1/6^2+\ldots)^{-1}$
  $= 6/\pi^2$
  ≈ 0.61

each positive number has a unique prime number factorization
ex. $45^2 = 3^4 \cdot 5^2$

How large is $\phi(n)$?

- $\phi(n)$ is the number of integers less than n that are relative prime to n
- $\phi(n)/n$ is the probability that a randomly chosen integer is relatively prime to n
- Therefore, $\phi(n) \approx n \cdot 6/\pi^2$
- P_n = Pr { n random numbers have no common factor }
  - n independent random numbers all have a given prime p as a factor is $1/p^n$
  - They do not all have p as a common factor $1 - 1/p^n$
  - $P_n = (1+1/2^n+1/3^n+1/4^n+1/5^n+1/6^n+\ldots)^{-1}$ is the Riemann zeta function $\zeta(n)$ http://mathworld.wolfram.com/RiemannZetaFunction.html
  - Ex. $n=4$, $\zeta(4) = \pi^4/90 \approx 0.92$

Euler's Theorem

true when n is prime

true even when $n = p^k$

✧ If gcd(a,n)=1 then $a^{\phi(n)} \equiv 1 \pmod{n}$

Proof: ✡ let S be the set of integers $1 \le x < n$, with gcd(x, n) = 1

✡ define $\psi(x) \equiv a \cdot x \pmod{n}$ be a mapping $\psi: S \to Z$

✡ $\forall x \in S$ and gcd(a, n) = 1, [if $\psi(x) \equiv a \cdot x \equiv 0 \pmod{n} \Rightarrow x \equiv 0 \pmod{n}$]

$\psi(x) \neq 0 \pmod{n}$

gcd(ψ(x), n) = 1 [gcd(a, n)=1 and gcd(x, n)=1 (no common prime factors)]

$\Rightarrow \forall x \in S, \psi(x) \in S$, i.e. $\psi: S \to S$

✡ $\forall$ x, y $\in$ S, 'if $x \neq y$ then $\psi(x) \not\equiv \psi(y) \pmod{n}$'

[if $\psi(x) \equiv \psi(y) \Rightarrow a \cdot x \equiv a \cdot y \Rightarrow x \equiv y$ since gcd(a, n) = 1]

✡ from the above two observations, $\forall x \in S$, $\psi(x)$ are distinct elements of S (i.e. $\{\psi(x) \mid \forall x \in S\}$ is S)

✡ $\prod_{x \in S} x \equiv \prod_{x \in S} \psi(x) \equiv a^{\phi(n)} \prod_{x \in S} x \pmod{n}$

✡ since gcd(x, n) = 1 for $x \in S$, we can cancel one by one $x \in S$ of both sides, and obtain $a^{\phi(n)} \equiv 1 \pmod{n}$

Euler's Theorem

✧ Example: What are the last three digits of 7^{803}?

i.e. we want to find $7^{803} \pmod{1000}$

$1000 = 2^3 \cdot 5^3$, $\phi(1000) = 1000(1-1/2)(1-1/5) = 400$

$7^{803} \equiv 7^{803 \pmod{400}} \equiv 7^3 \equiv 343 \pmod{1000}$)

✧ Example: Compute $2^{43210} \pmod{101}$?

$101 = 1 \cdot 101$, $\phi(101) = 100$

$2^{43210} \equiv 2^{43210 \pmod{100}} \equiv 2^{10} \equiv 1024 \equiv 14 \pmod{101}$

A second proof of Euler's Theorem

Euler's Theorem: $\forall a \in Z_n^*$, $a^{\phi(n)} \equiv 1 \pmod{n}$

✧ We have proved the above theorem by showing that the function $\psi(x) \equiv a \cdot x \pmod{n}$ is a permutation.

✧ We can also prove it through Fermat's Little Theorem & CRT

➢ consider $n = p \cdot q$, $\phi(n) = (p-1)(q-1)$

$\forall a \in Z_p^*$, $a^{p-1} \equiv 1 \pmod{p} \Rightarrow (a^{p-1})^{q-1} \equiv a^{\phi(n)} \equiv 1 \pmod{p}$

$\forall a \in Z_q^*$, $a^{q-1} \equiv 1 \pmod{q} \Rightarrow (a^{q-1})^{p-1} \equiv a^{\phi(n)} \equiv 1 \pmod{q}$

gcd(p,q)=1 $\Rightarrow$ $p \cdot q \mid a^{\phi(n)}-1$, i.e. $\forall a \in Z_n^*$ ($p \nmid a$ and $q \nmid a$), $\underline{a^{\phi(n)} \equiv 1 \pmod{n}}$

➢ consider $n = p^r$, $\phi(n) = p^{r-1}(p-1)$

$\forall a \in Z_{p^r}^*$, $a^{p-1} \equiv 1 \pmod{p} \Rightarrow a^{p-1} = 1 + \lambda p$

$a^{\phi(n)} \equiv (1+\lambda p)^{p^{r-1}} \equiv 1 \pmod{n}$

$a^{\phi(n)} = (1+\lambda p)^{p^{r-1}} = 1 + C_1^{p^{r-1}} \lambda p + C_2^{p^{r-1}} (\lambda p)^2 + \ldots$

$= 1 + p^{r-1} \lambda p + p^{r-1}(p^{r-1}-1)/2\, (\lambda p)^2 + \ldots$

A second proof (cont'd)

➢ consider $n = p^r \cdot q^s$, $\phi(n) = p^{r-1}(p-1)\, q^{s-1}(q-1)$

$\forall a \in Z_{p^r}^*$, $a^{p-1} \equiv 1 \pmod{p} \Rightarrow (a^{p-1})^{p^{r-1}} \equiv 1 \pmod{p^r}$

$\Rightarrow (a^{(p-1)p^{r-1}})^{(q-1)q^{s-1}} \equiv a^{\phi(n)} \equiv 1 \pmod{p^r} \Rightarrow p^r \mid a^{\phi(n)}-1$

$\forall a \in Z_{q^s}^*$, $a^{q-1} \equiv 1 \pmod{q} \Rightarrow (a^{q-1})^{q^{s-1}} \equiv 1 \pmod{q^s}$

$\Rightarrow (a^{(q-1)q^{s-1}})^{(p-1)p^{r-1}} \equiv a^{\phi(n)} \equiv 1 \pmod{q^s} \Rightarrow q^s \mid a^{\phi(n)}-1$

gcd(p^r,q^s)=1 $\Rightarrow p^r q^s \mid a^{\phi(n)}-1$, i.e. $\forall a \in Z_n^*$ ($p \nmid a$ and $q \nmid a$), $\underline{a^{\phi(n)} \equiv 1 \pmod{n}}$

➢ consider $n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$, $\phi(n) = n \prod_{\forall p \mid n} (1-1/p)$ **Unique Prime Factorization**

$\forall a \in Z_{p_i^{r_i}}^*$, $a^{p_i-1} \equiv 1 \pmod{p_i} \Rightarrow (a^{p_i-1})^{p_i^{r_i-1}} \equiv 1 \pmod{p_i^{r_i}}$

$\Rightarrow (a^{(p_i-1)p_i^{r_i-1}})^{\prod_{\forall j \neq i} (p_j-1)p_j^{r_j-1}} \equiv a^{\phi(n)} \equiv 1 \pmod{p_i^{r_i}} \Rightarrow p_i^{r_i} \mid a^{\phi(n)}-1$

all $p_i^{r_i}$ are relatively prime $\Rightarrow \prod_{i=1}^{k} p_i^{r_i} \mid a^{\phi(n)}-1$, i.e. $\forall a \in Z_n^*$ ($\forall i, p_i \nmid a$), $\underline{a^{\phi(n)} \equiv 1 \pmod{n}}$

Carmichael Theorem

Theorem:

$\forall a \in Z_n^*,\ a^{\lambda(n)} \equiv 1 \pmod n$ and $a^{n\cdot\lambda(n)} \equiv 1 \pmod{n^2}$

where $n=p\cdot q$, $p \neq q$, $\lambda(n) = \mathrm{lcm}(p-1, q-1)$, $\lambda(n) \mid \phi(n)$

✧ like Euler's Theorem, we can prove it through Fermat's Little Theorem, consider $n = p \cdot q$, where $p\neq q$,

$\forall a\in Z_p^*,\ a^{p-1} \equiv 1 \pmod p \Rightarrow (a^{p-1})^{(q-1)/\gcd(p-1,q-1)} \equiv a^{\lambda(n)} \equiv 1 \pmod p$

$\forall a\in Z_q^*,\ a^{q-1} \equiv 1 \pmod q \Rightarrow (a^{q-1})^{(p-1)/\gcd(p-1,q-1)} \equiv a^{\lambda(n)} \equiv 1 \pmod q$

$\gcd(p,q)=1 \Rightarrow pq \mid a^{\lambda(n)}-1,\ \forall a\in Z_n^*$ (i.e. $p\nmid a \wedge q\nmid a$), $a^{\lambda(n)} \equiv 1 \pmod n$

therefore, $\forall a\in Z_n^*,\ a^{\lambda(n)} = 1 + k \cdot n$

raise both side to the n-th power, we get $a^{n\cdot\lambda(n)} = (1 + k \cdot n)^n$,

$\Rightarrow a^{n\cdot\lambda(n)} = 1 + n\cdot k\cdot n + \ldots \Rightarrow \forall a \in Z_n^*$ (or $Z_{n^2}^*$), $a^{n\cdot\lambda(n)} \equiv 1 \pmod{n^2}$

Basic Principle to do Exponentiation

✧ Let a, n, x, y be integers with $n\geq 1$, and $\gcd(a,n)=1$ if $x \equiv y \pmod{\phi(n)}$, then $a^x \equiv a^y \pmod n$.

✧ If you want to work mod n, you should work mod $\phi(n)$ or $\lambda(n)$ in the exponent.

Primitive Roots modulo p

✧ When p is a prime number, a primitive root modulo p is a number whose powers yield every nonzero element mod p. (equivalently, the order of a primitive root is p-1)

✧ ex: $3^1\equiv3,\ 3^2\equiv2,\ 3^3\equiv6,\ 3^4\equiv4,\ 3^5\equiv5,\ 3^6\equiv1 \pmod 7$
3 is a primitive root mod 7

✧ sometimes called a multiplicative generator

✧ there are plenty of primitive roots, actually $\phi(p-1)$

- ex. $p=101$, $\phi(p-1)=100\cdot(1-1/2)\cdot(1-1/5)=40$
  $p=143537$, $\phi(p-1)=143536\cdot(1-1/2)\cdot(1-1/8971)=71760$

Primitive Testing Procedure

✧ How do we test whether h is a primitive root modulo p?

- naïve method:
  go through all powers $h^2, h^3, \ldots, h^{p-2}$, and make sure they all $\neq 1$ modulo p
- faster method:
  assume p-1 has prime factors $q_1, q_2, \ldots, q_n$,
  for all q_i, make sure $h^{(p-1)/q_i}$ modulo p is not 1,
  then h is a primitive root

Intuition: let $h \equiv g^a \pmod p$, if $\gcd(a, p-1)=d$ (i.e. g^a is not a primitive root), $(g^a)^{(p-1)/q_i} \equiv (g^{a/q_i})^{(p-1)} \equiv 1 \pmod p$ for some $q_i \mid d$

ex. $p=29$, $p-1=2\cdot2\cdot7$, $h=5$, $h^{28/2}=1$, $h^{28/7}=16$, 5 is not a primitive
$h=11$, $h^{28/2}=28$, $h^{28/7}=25$, 11 is a primitive

Primitive Testing Procedure (cont'd)

- Procedure to test a primitive g:

let p-1 has prime factors $q_1, q_2, \ldots, q_n$, (i.e. $\phi(p)=p-1=q_1^{r_1}\ldots q_n^{r_n}$)
for all q_i, $g^{(p-1)/q_i}$ (mod p) is not 1 $\Rightarrow$ g is a primitive

Proof:

(a) by definition, $ord_p(g)$ is the smallest positive x s.t. $g^x \equiv 1 \pmod p$
   Fermat Theorem: $g^{\phi(p)} \equiv 1 \pmod p$ therefore implies $ord_p(g) \le \phi(p)$
   if $\phi(p) = ord_p(g) * k + s$ with $0 \le s < ord_p(g)$
   $g^{\phi(p)} \equiv g^{ord_p(g) * k} g^s \equiv g^s \equiv 1 \pmod p$, but $s < ord_p(g) \Rightarrow s = 0$, i.e. $ord_p(g) \mid \phi(p)$

(b) assume g is not a primitive root i.e $ord_p(g) < \phi(p)=p-1$
   then $\exists$ i, such that $ord_p(g) \mid (p-1)/q_i$ i.e. $g^{(p-1)/q_i} \equiv 1 \pmod p$ for some q_i

(c) if for all q_i, $g^{(p-1)/q_i} \ne 1 \pmod p$
   then $ord_p(g) = \phi(p)$ and g is a primitive root modulo p

Lucas Primality Test

- An integer n is **prime** iff ***the converse of Fermat Little Theorem***

$\exists$a, s.t. **1.** $a^{n-1} \equiv 1 \pmod n$
2. $\forall$prime factor q of n-1, $a^{n-1/q} \ne 1 \pmod n$

Proof:

($\Rightarrow$) if n is prime, ***catch: inefficient, factors of n-1 are required***
   Fermat's little theorem ensures that "$\forall a \ne kn$, $a^{n-1} \equiv 1 \pmod n$"
   a primitive a ensures "$\forall$ prime factor q of n-1, $a^{n-1/q} \ne 1 \pmod n$"

($\Leftarrow$) if $\exists$a, s.t. 1. $a^{n-1} \equiv 1 \pmod n$ and
   2. $\forall$prime factor q of n-1, $a^{n-1/q} \ne 1 \pmod n$
   By definition, $ord_n(a)$ is the smallest positive x s.t. $a^x \equiv 1 \pmod n$
   the first condition implies that $ord_n(a) \le n-1$, also, $ord_n(a) \mid n-1$
   the second condition then implies that $ord_n(a) = n-1$ (*)
   Euler thm says that $a^{\phi(n)} \equiv 1 \pmod n$, by definition $\phi(n)<n-1$ if n is a composite number, i.e. $ord_n(a) < n-1$, contradict with (*).

Pratt's Primality Certificate

- Pratt's proved in 1975 that this polynomial-size structure can **prove that a number is prime** and is verifiable in polynomial time
- based on the **Lucas Primality Test (LPT)**
- example:

229 ($a = 6$, $229 - 1 = 2^2 \times 3 \times 19$)
   2 (known prime)
   3 ($a = 2$, $3 - 1 = 2$)
      2 (known prime)
   19 ($a = 2$, $19 - 1 = 2 \times 3^2$)
      2 (known prime)
      3 ($a = 2$, $3 - 1 = 2$)
         2 (known prime)

verification

$2^{3-1} \equiv 1 \pmod 3$
$2^{2/2} \equiv 2 \pmod 3$
By LPT, 2 is a prime, then 3 is also a prime

$2^{19-1} \equiv 1 \pmod{19}$
$2^{18/2} \equiv 18 \pmod{19}$
$2^{18/3} \equiv 7 \pmod{19}$
By LPT, if 2 and 3 are primes, then 19 is also a prime

$6^{229-1} \equiv 1 \pmod{229}$
$6^{228/2} \equiv 228 \pmod{229}$
$6^{228/3} \equiv 134 \pmod{229}$
$6^{228/19} \equiv 165 \pmod{229}$
By LPT, if 2, 3, 19 are primes, then 229 is also a prime

Number of Primitive Root in Z_p^*

- Why are there $\phi(p-1)$ primitive roots?
  - let g be a primitive root (the order of g is p-1)
  - $g, g^2, g^3, \ldots, g^{p-1}$ is a permutation of 1,2,…p-1 (an integer less than p-1)
  - if gcd(a, p-1)=d, then $(g^a)^{(p-1)/d} \equiv (g^{a/d})^{(p-1)} \equiv 1 \pmod p$ which says that the order of g^a is at most (p-1)/d, therefore, g^a is not a primitive root $\Rightarrow$ There are at most $\phi(p-1)$ primitive roots in Z_p^*
  - For an element g^a in Z_p^* where gcd(a, p-1) = 1, it is guaranteed that $(g^a)^{(p-1)/q_i} \ne 1 \pmod p$ for all q_i (q_i is factors or p-1)

   assume that for a certain q_i, $(g^a)^{(p-1)/q_i} \equiv 1 \pmod p$
   $\Rightarrow$ $p-1 \mid a \cdot (p-1) / q_i$
   $\Rightarrow$ $\exists$ integer k, $a \cdot (p-1) / q_i = k \cdot (p-1)$ i.e. $a = k \cdot q_i$
   $\Rightarrow$ $q_i \mid a$
   $\Rightarrow$ $q_i \mid \gcd(a, p-1)$ contradiction

Multiplicative Generators in Z_n^*

- How do we define a multiplicative generator in Z_n^* if n is a composite number?
  - Is there an element in Z_n^* that can generate all elements of Z_n^*?
  - If $n = p \cdot q$, the answer is negative. From Carmichael theorem, $\forall a \in Z_n^*$, $a^{\lambda(n)} \equiv 1 \pmod n$, gcd(p-1, q-1) is at least 2, $\lambda(n) = \text{lcm}(p-1, q-1)$ is at most $\phi(n) / 2$. The size of a maximal possible multiplicative subgroup in Z_n^* is therefore no larger than $\lambda(n)$.
  - If $n = p^k$, the answer is yes
  - How many elements in Z_n^* can generate the maximal possible subgroup of Z_n^*?

Finding Square Roots mod n

- For example: find x such that $x^2 \equiv 71 \pmod{77}$
  - Is there any solution?
  - How many solutions are there?
  - How do we solve the above equation systematically?
- In general: find x s.t. $x^2 \equiv b \pmod n$,

  where $b \in QR_n$, $n = p \cdot q$, and p, q are prime numbers
- Easier case: find x s.t. $x^2 \equiv b \pmod p$,

  where p is a prime number, $b \in QR_p$

Note: QR_n is "Quadratic Residue in Z_n^*" to be defined later

Finding Square Root mod *p*

- Given $y \in Z_p^*$, find x, s.t. $x^2 \equiv y \pmod p$, p is prime

Two cases:
  - $p \equiv 1 \pmod 4$ (i.e. $p = 4k + 1$) : probabilistic algorithm
  - $p \equiv 3 \pmod 4$ (i.e. $p = 4k + 3$) : deterministic algorithm
- Is there any solution? (Is y a QR_p?)

  check $\quad y^{\frac{p-1}{2}} \overset{?}{\equiv} 1 \pmod p$ ***Euler's Criterion***
- $p \equiv 3 \pmod 4$

$$x \equiv \pm y^{\frac{p+1}{4}} \pmod p$$

  - $(p+1)/4 = (4k+3+1)/4 = k+1$ is an integer
  - $x^2 = y^{(p+1)/2} = y^{(p-1)/2} \cdot y \equiv y \pmod p$

Finding Square Root mod *p*

- $p \equiv 1 \pmod 4$
  - Peralta, Eurocrypt'86, $p = 2^s q + 1$, both p, q are prime
  - 3-step probabilistic procedure
    1. Choose a random number r, if $r^2 \equiv y \pmod p$, output $z = r$
    2. Calculate $(r + x)^{(p-1)/2} \equiv u + v\,x \pmod{f(x)}$, $\quad f(x) = x^2 - y$
    3. If $u = 0$ then output $z \equiv v^{-1} \pmod p$, else goto step 1

  note: $(b+cx)(d+ex) \equiv (bd+ce\,x^2) + (be+cd)\,x$

  $\equiv (bd+ce\,y) + (be+cd)\,x \pmod{x^2-y}$

  use *square-multiply* algorithm to calculate the polynomial $(r + x)^{(p-1)/2}$
  - the probability to successfully find z for each $r \geq 1/2$

Finding Square Root mod p

✧ ex: find z such that $z^2 \equiv 12 \pmod{13}$

solution:

- ✡ $13 \equiv 1 \pmod 4$ ie. 4k+1
- ✡ choose $r = 3$, $3^2 = 9 \neq 12$
- ✡ $(3 + x)^{(13-1)/2} = (3 + x)^6 \equiv 12 + 0\,x \pmod{x^2-12}$
- ✡ choose $r = 7$, $7^2 \equiv 10 \neq 12$
- ✡ $(7 + x)^{(13-1)/2} = (7 + x)^6 \equiv 0 + 8\,x \pmod{x^2-12}$
  $\Rightarrow z = 8^{-1} = 5 \pmod{13}$

Why does it work???

Why is the success probability > ½ ???

Finding Square Roots mod n

✧ Now let's return to the question of solving square roots in Z_n^*, i.e.

for an integer $y \in QR_n$,

find $x \in Z_n^*$ such that $x^2 \equiv y \pmod n$

✧ We would like to transform the problem into solving square roots mod p.

✧ Question: for $n=p\cdot q$

Is solving "$x^2 \equiv y \pmod n$" equivalent to solving "$x^2 \equiv y \pmod p$ and $x^2 \equiv y \pmod q$"???

yes
($\Rightarrow$) $x^2-y=kn=kpq \Rightarrow p \mid x^2-y$ and $q \mid x^2-y$ □
($\Leftarrow$) $p \mid x^2-y$ and $q \mid x^2-y \Rightarrow pq \mid x^2-y$ i.e. $x^2-y=kpq=kn$ □

Finding Square Roots mod p·q

✧ find x such that $x^2 \equiv 71 \pmod{77}$

- ✶ $77 = 7 \cdot 11$
- ✶ "x^* satisfies $f(x^*) \equiv 71 \pmod{77}$" $\Leftrightarrow$
  "x^* satisfies both $f(x^*) \equiv 1 \pmod 7$ and $f(x^*) \equiv 5 \pmod{11}$"
- ✶ since 7 and 11 are prime numbers, we can solve $x^2 \equiv 1 \pmod 7$ and $x^2 \equiv 5 \pmod{11}$ far more easily than $x^2 \equiv 71 \pmod{77}$
  - $x^2 \equiv 1 \pmod 7$ has two solutions: $x \equiv \pm 1 \pmod 7$
  - $x^2 \equiv 5 \pmod{11}$ has two solutions: $x \equiv \pm 4 \pmod{11}$
- ✶ put them together and use CRT to calculate the four solutions
  - $x \equiv 1 \pmod 7 \equiv 4 \pmod{11} \Rightarrow x \equiv 15 \pmod{77}$
  - $x \equiv 1 \pmod 7 \equiv 7 \pmod{11} \Rightarrow x \equiv 29 \pmod{77}$
  - $x \equiv 6 \pmod 7 \equiv 4 \pmod{11} \Rightarrow x \equiv 48 \pmod{77}$
  - $x \equiv 6 \pmod 7 \equiv 7 \pmod{11} \Rightarrow x \equiv 62 \pmod{77}$

Computational Equivalence to Factoring

✧ Previous slides show that once you know the factors of n are p and q, you can easily solve the square roots of n

✧ Indeed, if you can solve the square roots for one single quadratic residue mod n, you can factor n.

- ✶ from the four solutions $\pm a$, $\pm b$ on the previous slide
  - $x \equiv c \pmod p \equiv d \pmod q \Rightarrow x \equiv a \pmod{p\cdot q}$
  - $x \equiv c \pmod p \equiv -d \pmod q \Rightarrow x \equiv b \pmod{p\cdot q}$
  - $x \equiv -c \pmod p \equiv d \pmod q \Rightarrow x \equiv -b \pmod{p\cdot q}$
  - $x \equiv -c \pmod p \equiv -d \pmod q \Rightarrow x \equiv -a \pmod{p\cdot q}$

  we can find out $a \equiv b \pmod p$ and $a \equiv -b \pmod q$
  (or equivalently $a \equiv -b \pmod p$ and $a \equiv b \pmod q$)
- ✶ therefore, $p \mid (a-b)$ i.e. $\gcd(a-b, n) = p$ (ex. gcd(15-29, 77)=7)
  $q \mid (a+b)$ i.e. $\gcd(a+b, n) = q$ (ex. gcd(15+29, 77)=11)

Quadratic Residues

- Consider $y \in Z_n^*$, if $\exists\, x \in Z_n^*$, such that $x^2 \equiv y \pmod{n}$, then y is called a quadratic residue mod n, i.e. $y \in QR_n$
- If the modulus p is prime, there are $(p\text{-}1)/2$ quadratic residues in Z_p^*
  - let g be a primitive root in Z_p^*, $\{g, g^2, g^3, \ldots, g^{p-1}\}$ is a permutation of $\{1,2,\ldots p\text{-}1\}$
  - in the above set, $\{g^2, g^4, \ldots, g^{p-1}\}$ are quadratic residues (QR_p)
  - $\{g, g^3, \ldots, g^{p-2}\}$ are quadratic non-residues (QNR_p), out of which there are $\phi(p\text{-}1)$ primitive roots

Quadratic Residues in Z_p^*

1st proof:

- For each $x \in Z_p^*$, $p\text{-}x \neq x \pmod{p}$ (since if x is odd, $p\text{-}x$ is even), it's clear that x and $p\text{-}x$ are both square roots of a certain $y \in Z_p^*$
- Because there are only $p\text{-}1$ elements in Z_p^*, we know that $|QR_p| \leq (p\text{-}1)/2$
- Because $|\,\{g^2, g^4, \ldots, g^{p-1}\}\,| = (p\text{-}1)/2$, there can be no more quadratic residues outside this set. Therefore, the set $\{g, g^3, \ldots, g^{p-2}\}$ contains only quadratic non-residues

Quadratic Residues in Z_p^*

2nd proof:

- Because the squares of x and $p\text{-}x$ are the same, the number of quadratic residues must be less than $p\text{-}1$ (i.e. some element in Z_p^* must be quadratic non-residue)
- Let g is a primitive, consider this set $\{g, g^3, \ldots, g^{p-2}\}$ directly
- If $g \in QR_p$, then g cannot be a primitive (because g^k must all be quadratic residues). Thus, $g \in QNR_p$
- If $g^{2k+1} \equiv g^{2k} \cdot g \in QR_p$, $\exists\, x \in Z_p^*$ such that $x^2 \equiv g^{2k} \cdot g \pmod{p}$

  Since $\gcd(g^{2k}, p)=1$, $g \equiv (g^{2k})^{-1} \cdot x^2 \equiv ((g^{-1})^k \cdot x)^2 \in QR_p$ contradiction

  Thus, $g^{2k+1} \in QNR_p$

  $(g^{2k})^{-1}(g^{2k}) \equiv (g^{2k})^{-1} g \cdot g \cdot \ldots \cdot g \equiv 1 \pmod{p}$
  $\Rightarrow (g^{2k})^{-1} \equiv g^{-1} \cdot g^{-1} \cdot \ldots \cdot g^{-1} \equiv (g^{-1})^{2k} \equiv ((g^{-1})^k)^2$

Quadratic Residues in Z_p^*

- ex. p=143537, p-1=143536=$2^4 \cdot 8971$,

  $\phi(p\text{-}1) = 2^4 \cdot 8971 \cdot (1\text{-}1/2) \cdot (1\text{-}1/8971)$=71760 primitives,

  $(p\text{-}1)/2$=71768 QR_p's and 71768 QNR_p's
  - Note: if g is a primitive, then g^3, g^5 … are also primitives except the following 8 numbers $g^{8971}, g^{8971 \cdot 3}, \ldots, g^{8971 \cdot 15}$
  - Elements in Z_p^* can be grouped further according to their order since $\forall x \in Z_p^*$, $ord_p(x) \mid p\text{-}1$, we can list all possible orders

$ord_p(x)$	p-1	$\frac{p-1}{2}$	$\frac{p-1}{4}$	$\frac{p-1}{8}$	8971 $\frac{p-1}{16}$	16 $\frac{p-1}{8971}$	8 $\frac{p-1}{8971 \cdot 2}$	4 $\frac{p-1}{8971 \cdot 4}$	2 $\frac{p-1}{8971 \cdot 8}$	1 $\frac{p-1}{8971 \cdot 16}$
	QNR_p	QR_p	QR_p	QR_p	QR_p	QNR_p	QR_p	QR_p	QR_p	QR_p
#	$\phi(p\text{-}1)$					8			2	1

QR_n for Composite Modulus n

✧ If y is a quadratic residue modulo n, it must be a quadratic residue modulo all prime factors of n.

$\exists\, x \in Z_n^*$ s.t. $x^2 \equiv y \pmod{n} \Leftrightarrow x^2 = k\cdot n + y = k\cdot p\cdot q + y$

$\Rightarrow x^2 \equiv y \pmod{p}$ and $x^2 \equiv y \pmod{q}$

✧ If y is a quadratic residue modulo p and also a quadratic residue modulo q, then y is a quadratic residue modulo n.

$\exists\, r_1 \in Z_p^*$ and $r_2 \in Z_q^*$ such that

$y \equiv r_1^2 \pmod{p} \equiv (r_1 \bmod p)^2 \pmod{p}$

$\equiv r_2^2 \pmod{q} \equiv (r_2 \bmod q)^2 \pmod{q}$

from CRT, $\exists!\, r \in Z_n^*$ such that $r \equiv r_1 \pmod{p} \equiv r_2 \pmod{q}$

therefore, $y \equiv r^2 \pmod{p} \equiv r^2 \pmod{q}$

again from CRT, $y \equiv r^2 \pmod{p\cdot q}$

Legendre Symbol

✧ Legendre symbol $L(a, p)$ is defined when a is any integer, p is a prime number greater than 2

- ✶ $L(a, p) = 0$ if $p \mid a$
- ✶ $L(a, p) = 1$ if a is a quadratic residue mod p
- ✶ $L(a, p) = -1$ if a is a quadratic non-residue mod p

✧ Two methods to compute (a/p)

- ✶ $(a/p) = a^{(p-1)/2} \pmod{p}$
- ✶ recursively calculate by $L(a \cdot b, p) = L(a, p) \cdot L(b, p)$
  1. If $a = 1$, $L(a, p) = 1$
  2. If a is even, $L(a, p) = L(a/2, p)\cdot(-1)^{(p^2-1)/8}$
  3. If a is odd prime, $L(a, p) = L((p \bmod a), a)\cdot(-1)^{(a-1)(p-1)/4}$

✧ Legendre symbol $L(a, p) = -1$ if $a \in QNR_p$

$L(a, p) = 1$ if $a \in QR_p$

Legendre Symbol

$$y \in QR_p \Leftrightarrow y^{(p-1)/2} \equiv 1 \pmod{p}$$

$(\Rightarrow)$

- ✶ If $y \in QR_p$
- ✶ Then $\exists x \in Z_p^*$ such that $y \equiv x^2 \pmod{p}$
- ✶ Therefore, $y^{(p-1)/2} \equiv (x^2)^{(p-1)/2} \equiv x^{(p-1)} \equiv 1 \pmod{p}$

$(\Leftarrow)$

- ✶ If $y \notin QR_p$ i.e. $y \in QNR_p$
- ✶ Then $y \equiv g^{2k+1} \pmod{p}$
- ✶ Therefore, $y^{(p-1)/2} \equiv (g^{2k} \cdot g)^{(p-1)/2} \equiv g^{k(p-1)}\, g^{(p-1)/2} \equiv g^{(p-1)/2} \not\equiv 1 \pmod{p}$

$ord_p(g) = p-1$

Jacobi Symbol

✧ Jacobi symbol $J(a, n)$ is a generalization of the Legendre symbol to a composite modulus n

✧ If n is a prime, $J(a, n)$ is equal to the Legendre symbol i.e. $J(a, n) \equiv a^{(n-1)/2} \pmod{n}$

✧ Jacobi symbol cannot be used to determine whether a is a quadratic residue mod n (unless n is a prime)

ex. $J(7, 143) = J(7, 11)\cdot J(7, 13) = (-1)\cdot(-1) = 1$

however, there is no integer x such that

$x^2 \equiv 7 \pmod{143}$

Calculation of Jacobi Symbol

✧ The following algorithm computes the Jacobi symbol J(a, n), for any integer a and odd integer n, recursively:

- Def 1: J(0, n) = 0 also If n is prime, J(a, n) = 0 if $n|a$
- Def 2: If n is prime, J(a, n) = 1 if $a \in QR_n$ and J(a, n) = -1 if $a \notin QR_n$
- Def 3: If n is a composite, J(a, n) = J(a, $p_1 \cdot p_2 \ldots \cdot p_m$) = J($a$,$p_1$)·J($a$,$p_2$)…·J($a$,$p_m$)
- Rule 1: J(1, n) = 1
- Rule 2: J($a \cdot b$, n) = J(a, n) · J(b, n)
- Rule 3: J(2, n) = 1 if $(n^2-1)/8$ is even and J(2, n) = -1 otherwise
- Rule 4: J(a, n) = J(a mod n, n)
- Rule 5: J(a, b) = J(-a, b) if a <0 and (b-1)/2 is even,
  J(a, b) = -J(-a, b) if a<0 and (b-1)/2 is odd
- Rule 6: J(a, $b_1 \cdot b_2$) = J(a, b_1) · J(a, b_2)
- Rule 7: if gcd(a, b)=1, a and b are odd
  - 7a: J(a, b) = J(b, a) if (a-1)·(b-1)/4 is even
  - 7b: J(a, b) = -J(b, a) if (a-1)·(b-1)/4 is odd

QR_n and Jacobi Symbol

✧ Consider $n = p \cdot q$, where p and q are prime numbers

$x \in QR_n$

$\Leftrightarrow x \in QR_p$ and $x \in QR_q$

$\Leftrightarrow J(x, p) = x^{(p-1)/2} \equiv 1 \pmod{p}$ and $J(x, q) = x^{(q-1)/2} \equiv 1 \pmod{q}$

$\Rightarrow J(x, n) = J(x, p) \cdot J(x, q) = 1$

	J(x, p)	J(x, q)	J(x, n)	
Q_{00}	1	1	1	$x \in QR_n$
Q_{01}	1	-1	-1	$x \in QNR_n$
Q_{10}	-1	1	-1	$x \in QNR_n$
Q_{11}	-1	-1	1	$x \in QNR_n$

Wilson's Theorem

$(p-1)! \equiv -1 \pmod{p}$

Proof:

Goal: $(p-1)! \equiv 1 \cdot 2 \cdot 3 \cdots (p-1) \equiv -1 \equiv (p-1) \pmod{p}$

- Since gcd(p-1, p) = 1, the above is equivalent to $(p-2)! \equiv 1 \pmod{p}$
- e.g. $p = 5$, $3 \cdot 2 \cdot 1 \equiv 1 \pmod{5}$
  $p = 7$, $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \equiv 1 \pmod{7}$
- We know that $1^{-1} \equiv 1 \pmod{p}$ and $(-1)^{-1} \equiv -1 \pmod{p}$
- Claim: $\forall i \in Z_p^* \backslash \{1,-1\}$, $i^{-1} \neq i$ (pf: if $i^{-1} \equiv i$ then $i^2 \equiv 1$, $i \in \{1,-1\}$)
- Claim: $\forall i_1 \neq i_2 \in Z_p^* \backslash \{1,-1\}$, $i_1^{-1} \neq i_2^{-1}$ (pf: if $i_1^{-1} \equiv i_2^{-1}$ then $i_1 \cdot i_2^{-1} \equiv 1$ then $i_1 \equiv i_2$, contradiction)
- Out of the set {2, 3, … p-2}, we can form (p-3)/2 pairs such that $i \cdot j \equiv 1$ (mod p), multiply them together, we obtain $(p-2)! \equiv 1$

Another Proof of QR_p test

$$y \in QR_p \Leftrightarrow y^{(p-1)/2} \equiv 1 \pmod p$$

($\Rightarrow$)
- ✶ If $y \in QR_p$
- ✶ Then $\exists x \in Z_p^*$ such that $y \equiv x^2 \pmod p$
- ✶ Therefore, $y^{(p-1)/2} \equiv (x^2)^{(p-1)/2} \equiv x^{(p-1)} \equiv 1 \pmod p$

($\Leftarrow$)
- ✶ $\forall i, y \in Z_p^*$, $\gcd(i, p)=1$, $\exists j$ such that $i \cdot j \equiv y \pmod p$
- ✶ If $y \notin QR_p$, the congruence $x^2 \equiv y \pmod p$ has no solution, therefore, $j \neq i \pmod p$
- ✶ We can group the integers 1, 2, …, p-1 into (p-1)/2 pairs (i, j), each satisfying $i \cdot j \equiv y \pmod p$
- ✶ Multiply them together, we have $(p-1)! \equiv y^{(p-1)/2} \pmod p$
- ✶ From Wilson's theorem, $y^{(p-1)/2} \equiv -1 \pmod p$

Exactly Two Square Roots

Every $y \in QR_p$ has exactly two square roots i.e. x and p-x such that $x^2 \equiv (p-x)^2 \equiv y \pmod p$

pf:
- ✶ $QR_p = \{g^2, g^4, \ldots, g^{p-1}\}$, $|Z_p^*| = p-1$, and $|QR_p| = (p-1)/2$
- ✶ For each $y \equiv g^{2k}$ in QR_p, there are at least two distinct $x \in Z_p^*$ s.t. $x^2 \equiv y \pmod p$, i.e., g^k and p-g^k (if one is even, the other is odd)
- ✶ Since $|QR_p| = (p-1)/2$, we can obtain a set of p-1 square roots $S=\{g, p-g, g^2, p-g^2, \ldots, g^{(p-1)/2}, p-g^{(p-1)/2}\}$
- ✶ **Claim**: the elements of S are all distinct (1. $g^i \neq g^j \pmod p$ when $i \neq j$ since g is a primitive, 2. $g^i \neq -g^j \pmod p$ when $i \neq j$, otherwise $(g^i+g^j)(g^i-g^j) \equiv g^{2i}-g^{2j} \equiv 0$ (mod p) implies $i \neq j \pmod{(p-1)/2}$, 3. $g^i \neq -g^i \pmod p$ since if one is even, the other is odd)
- ✶ If there is one more square root z of $y \equiv g^{2k}$ which is not g^k and $-g^k$, it must belong to S (which is Z_p^*), say g^j, $j \neq k$, which would imply that $g^{2j} \equiv g^{2k} \pmod p$, and leads to contradiction

Order q Subgroup G_q of Z_p^*

- ✧ Let p be a prime number, g be a primitive in Z_p^*
- ✧ Let $p = k \cdot q + 1$ i.e. $q \mid p-1$ where q is also a prime number
- ✧ Let $G_q = \{g^k, g^{2k}, \ldots, g^{q \cdot k} \equiv 1\}$
- ✧ Is G_q a subgroup in Z_p^*? YES

  $\forall x, y \in G_q$, it is clear that $z \equiv g^{i \cdot k} \equiv x \cdot y \equiv g^{(i_1+i_2) \cdot k} \pmod p$ is also in G_q, where $i \equiv i_1 + i_2 \pmod q$
- ✧ Is the order of the subgroup G_q q? YES

  $\forall i_1, i_2 \in Z_q, i_1 \neq i_2$, $g^{i_1 \cdot k} \neq g^{i_2 \cdot k} \pmod p$ otherwise g is not a primitive in Z_p^*, also $g^{q \cdot k} \equiv 1 \pmod p$
- ✧ How many generators are there in G_q? $\phi(q)=q-1$

  a. there are $\phi(p-1)$ generators in $Z_p^*=\{g^1, g^2, \ldots, g^x, \ldots, g^{p-1}\}$, since $\gcd(p-1, x) = d > 1$ implies that $ord_p(g^x) = (p-1)/d$

Order q Subgroup G_q (cont'd)

also $(g^x)^y \equiv 1 \pmod p$ and $g^{p-1} \equiv 1 \pmod p$ implies that either $x \cdot y \mid p-1$ or $p-1 \mid x \cdot y$, $\gcd(x, p-1) = 1$ implies that $p-1 \mid y$ therefore, $ord_p(g^x) = p-1$

b. there are $\phi(q)$ primitives in $G_q = \{g^k, g^{2k}, \ldots, g^{q \cdot k} \equiv 1\}$ since q is also a prime number

- ✧ **Is G_q a unique order q subgroup in Z_p^* ?** YES

  Let S be an order-q cyclic subgroup, $S=\{g, g^2, \ldots, g^q \equiv 1\}$. Since p is prime, $\exists$ a unique k-th root $g_1 \in Z_p^*$, s.t. $g \equiv g_1^k \pmod p$

  Let $g_1 \neq g$ be another primitive, clearly $g_1 \equiv g^s \pmod p$,

  Is the set $S=\{g_1^k, g_1^{2k}, \ldots, g_1^{q \cdot k} \equiv 1\}$ different from G_q?

  let $x \in S$, i.e. $x \equiv g_1^{i_1 \cdot k} \pmod p$, $i_1 \in Z_q$

  $x \equiv g_1^{i_1 \cdot k} \equiv g^{s \cdot i_1 \cdot k} \equiv g^{i \cdot k} \pmod p$ where $i \equiv s \cdot i_1 \pmod q$, i.e. $S \subseteq G_q$

  The proof is similar for $G_q \subseteq S$. Therefore, $S = G_q$

Gauss' Lemma

Lemma: let p be a prime, a is an integer s.t. gcd(a, p)=1,

define $\{\alpha_j \equiv j\cdot a \pmod p\}_{j=1,\dots,(p-1)/2}$,

let n be the number of α_j's s.t. $\alpha_j > p/2$ then $L(a, p) = (-1)^n$

pf.

- $\alpha_j \in \{r_1, \dots, r_n\}$ if $\alpha_j > p/2$ and $\alpha_j \in \{s_1, \dots, s_{(p-1)/2-n}\}$ if $\alpha_j < p/2$
- Since gcd(a, p)=1, r_i and s_i are all distinct and non-zero
- Clearly, $0 < p-r_i < p/2$ for i=1,…,n
- no $p-r_i$ is an s_j: if $p-r_i=s_j$ then $s_j \equiv -r_i \pmod p$

  rewrite in terms of a: $u\,a \equiv -v\,a \pmod p$ where $1 \le u, v \le (p-1)/2$

  $\Rightarrow u \equiv -v \pmod p$ where $1 \le u, v \le (p-1)/2 \Rightarrow$ impossible

$\Rightarrow \{s_1, \dots, s_{(p-1)/2-n}, p-r_1, \dots, p-r_n\}$ is a reordering of $\{1, 2, \dots, (p-1)/2\}$

- Thus, $((p-1)/2)! \equiv s_1\cdots s_{(p-1)/2-n}\cdot(-r_1)\cdots(-r_n) \equiv (-1)^n\, s_1\cdots s_{(p-1)/2-n}\cdot r_1\cdots r_n$

  $\equiv (-1)^n\, ((p-1)/2)!\, a^{(p-1)/2} \pmod p \Rightarrow L(a, p) = (-1)^n$

Theorem: $J(2, p) = (-1)^{(p^2-1)/8}$

Theorem: let p be a prime, gcd(a, p) = 1 then $L(a, p) = (-1)^t$

where $t = \sum_{j=1}^{(p-1)/2} \lfloor j\cdot a/p \rfloor$. Also $L(2, p) = (-1)^{(p^2-1)/8}$

pf.

- $\alpha_j \in \{r_1, \dots, r_n\}$ if $\alpha_j > p/2$ and $\alpha_j \in \{s_1, \dots, s_{(p-1)/2-n}\}$ if $\alpha_j < p/2$
- $j\,a = p\lfloor j\cdot a/p \rfloor + \alpha_j$ for j=1, …, (p-1)/2

  $$\Rightarrow \sum_{j=1}^{(p-1)/2} j\,a = \sum_{j=1}^{(p-1)/2} p\lfloor j\cdot a/p \rfloor + \sum_{j=1}^{n} r_j + \sum_{j=1}^{(p-1)/2-n} s_j$$
- $\{s_1, \dots, s_{(p-1)/2-n}, p-r_1, \dots, p-r_n\}$ is a reordering of $\{1, 2, \dots, (p-1)/2\}$

  $$\Rightarrow \sum_{j=1}^{(p-1)/2} j = \sum_{j=1}^{n} (p-r_j) + \sum_{j=1}^{(p-1)/2-n} s_j = np - \sum_{j=1}^{n} r_j + \sum_{j=1}^{(p-1)/2-n} s_j$$
- Subtracting the above two equations, we have

  $$(a-1)\sum_{j=1}^{(p-1)/2} j = p\left(\sum_{j=1}^{(p-1)/2} \lfloor j\cdot a/p \rfloor - n\right) + 2\sum_{j=1}^{n} r_j$$

$J(2, p) = (-1)^{(p^2-1)/8}$ (cont'd)

- $\sum_{j=1}^{(p-1)/2} j = 1 + \dots + (p-1)/2 = (p-1)/2\,(1 + (p-1)/2)/2 = (p^2-1)/8$
- Thus, we have $(a-1)(p^2-1)/8 \equiv \sum_{j=1}^{(p-1)/2} \lfloor j\cdot a/p \rfloor - n \pmod 2$

- If a is odd, $n \equiv \sum_{j=1}^{(p-1)/2} \lfloor j\cdot a/p \rfloor$
- If a = 2, $\lfloor j\cdot 2/p \rfloor = 0$ for j=1, …, (p-1)/2, $n \equiv (p^2-1)/8 \pmod 2$

  therefore, $J(2, p) = (-1)^{(p^2-1)/8}$

Lemma. ord-k elements in $Z_p^* \le \phi(k)$

Lemma. There are at most $\phi(k)$ ord-k elements in Z_p^*, k | p-1

pf. ✧ Z_p^* is a field $\Rightarrow x^k-1 \equiv 0 \pmod p$ has at most k roots

✧ if ***a*** is a nontrivial root ($a \ne 1$), then $\{a^0, a^1, a^2, \dots, a^{k-1}\}$ is the set of the k distinct roots.

✧ Those a^ℓ with $\gcd(\ell, k) = d > 1$ have order at most **k/d**

✧ Only those a^ℓ with $\gcd(\ell, k) = 1$ might have order k

✧ Hence, there are at most $\phi(k)$ order k elements

e.g. p = 13 {2, 4, 8, 3, 6, 12, 11, 9, 5, 10, 7, 1}

2 is a generator in $Z_{13}^* = \{2^1, 2^2, 2^3, 2^4, 2^5, 2^6, 2^7, 2^8, 2^9, 2^{10}, 2^{11}, 2^{12}\}$

k=12, {2, ~~4~~, ~~8~~, ~~3~~, 6, ~~12~~, 11, ~~9~~, ~~5~~, ~~10~~, 7, ~~1~~}, $\phi(12)$

k=6, {4, ~~3~~, ~~12~~, ~~9~~, 10, ~~1~~}, $\phi(6)$ ← $(2^{(p-1)/k})^j = (2^2)^j$ k=3, {3, 9, ~~1~~}, $\phi(3)$

k=4, {8, ~~12~~, 5, ~~1~~}, $\phi(4)$ k=2, {12, ~~1~~}, $\phi(2)$ k=1, {1}, $\phi(1)$

Lemma. $\Sigma_{k|p-1}\ \phi(k) = p-1$

Lemma. $\Sigma_{k|p-1}\ \phi(k) = p-1$ let $\phi(1)=1$

pf.

$p-1 = \Sigma_{k|p-1}$ (# a in Z_p^* s.t. gcd(a, p-1) = k)

$= \Sigma_{k|p-1}$ (# b in $\{1,\ldots,(p-1)/k\}$ s.t. gcd(b, (p-1)/k) = 1) (b = a/k)

$= \Sigma_{k|p-1}\ \phi((p-1)/k)$

$= \Sigma_{k|p-1}\ \phi(k)$

$\{\phi(1), \phi(2), \phi(3), \phi(4), \phi(6), \phi(12)\}$

let p=13, $a \in Z_p^*$

gcd(a, p-1)=k $\Rightarrow$ k | p-1

k=1, {1,5,7,11}, $\phi(12/1)$

k=2, {2,10}, $\phi(12/2)$

k=3, {3,9}, $\phi(12/3)$

k=4, {4,8}, $\phi(12/4)$

k=6, {6}, $\phi(12/6)$

k=12, {12}, $\phi(12/12)$

Z_p^* is a *cyclic* group

Theorem: Z_p^* is a *cyclic* group for a prime number p

pf. Lemma 1: # of ord-k elements in $Z_p^* \le \phi(k)$, where k | p-1

Lemma 2: $\Sigma_{k|p-1}\ \phi(k) = p-1$

The order k of every element in Z_p^* divides p-1

$\Rightarrow \Sigma_{k|p-1}$ (# of elements in Z_p^* with order k) = p-1

(Lemma 1) $\Rightarrow p-1 \le \Sigma_{k|p-1}\ \phi(k)$, combined with lemma 2,

we know that # of ord-k elements in $Z_p^* = \phi(k)$

$\Rightarrow$ # of ord-(p-1) elements in $Z_p^* = \phi(p-1) > 1$

$\Rightarrow$ There is at least one generator in Z_p^*, i.e. Z_p^* is cyclic

Ex. p=13, p-1 = |{2,6,11,7}| + |{4,10}| + |{8,5}| + |{3,9}| + |{12}| + |{1}|
k=12 k=6 k=4 k=3 k=2 k=1

Generators in QR_n

- Number of generators in Z_p^*: $\phi(p-1)$

  Let g be a primitive, $Z_p^* = \langle g\rangle = \{g, g^2, g^3, \ldots, g^k, \ldots, g^{p-1}\}$

  if gcd(k, p-1) = d ≠ 1 then g^k is not a primitive

  since $(g^k)^{(p-1)/d} = (g^{k/d})^{p-1} = 1$, i.e. $ord_p(g^k) \le (p-1)/d$

  if gcd(k, p-1) = 1 and g^k is not a primitive, then $d=ord_p(g^k) < p-1$, i.e.

  $(g^k)^d = 1$; g is a primitive $\Rightarrow$ p-1 | k d $\Rightarrow$ p-1 | d contradiction.
- Z_n^* is not a cyclic group (n = p q, p=2p'+1, q=2q'+1, $\lambda(n)$=2p'q')

  Since $x^{\lambda(n)} \equiv 1 \pmod n$, there is no generator that can generate all members in Z_n^*
- QR_n is a cyclic group of order $\lambda(n)/2$ = lcm(p-1, q-1)/2 = p' q'

  $\forall x \in Z_n^*$, $x^{\lambda(n)} \equiv 1 \pmod n$ Carmichael's Theorem

  clearly, $(x^2)^{\lambda(n)/2} \equiv 1 \pmod n$, $QR_n = \{x^2 \mid \forall x \in Z_n^*\}$

  i.e. $\forall y \in QR_n$, $ord_n(y)$ | p' q' ($ord_n(y) \in \{1, p', q', p'q'\}$)

Generators in QR_n (cont'd)

cyclic? $\exists x^* \in Z_n^*$ $ord_n(x^*) = \lambda(n) = 2p'q' \Rightarrow$

$\exists y^* (=(x^*)^2) \in QR_n$ s.t. $ord_n(y^*) = \lambda(n)/2 = p'q'$

- Let y be a random element in QR_n, the probability that y is a generator is close to 1

  Let y^* be a generator of QR_n,

  $QR_n = \langle y^*\rangle = \{y^*, (y^*)^2, (y^*)^3, \ldots, (y^*)^k, \ldots, (y^*)^{p'q'}\}$

  if gcd(k, p'q') = d ≠ 1 then $(y^*)^k$ is not a generator

  since $((y^*)^k)^{p'q'/d} = ((y^*)^{k/d})^{p'q'} = 1$, i.e. $ord_p((y^*)^k) \le (p'q')/d$

  $\phi(p'q') = \phi(p')\,\phi(q') = (p'-1)(q'-1) = p'q' - p' - q' + 1$

  $= p'q' - (p'-1) - (q'-1) - 1$

  $\forall x \in \{(y^*)^{q'}, (y^*)^{2q'}, \ldots, (y^*)^{(p'-1)q'}\}$ $ord_n(x) = p'$

  $\forall x \in \{(y^*)^{p'}, (y^*)^{2p'}, \ldots, (y^*)^{(q'-1)p'}\}$ $ord_n(x) = q'$

  $ord_n(1) = 1$

  Pr{x is a generator | $x \in_R QR_n$} = $\phi(p'q')$ / (p'q') is close to 1

Subgroups in Z_n^*

Consider n = p q, p=2p'+1, q=2q'+1, m=p'q', $\lambda(n)$ = lcm(p-1, q-1)=2m,

$\phi(n) = (p-1)(q-1) = 4m$

- ✧ $\mathbf{Z_n^*}$ is not a cyclic group
  - ✶ Carmichael's theorem asserts that no element in Z_n^* can generate all elements in Z_n^*. (maximum order is 2m instead of 4m)
  - ✶ However, Z_n^* is still a group over modulo n multiplication.
- ✧ $\mathbf{QR_n}$ is a cyclic subgroup of order $m = \lambda(n)/2$, $QR_n = \{x^2 \mid \forall x \in Z_n^*\}$
  - ✶ $J_{00} = \{x \in Z_n^* \mid J(x,p)=1 \text{ and } J(x,q)=1\}$
  - ✶ If there exists an element in Z_n^* whose order is 2m, then QR_n is clearly a cyclic group. (Will the precondition be true?)
  - ✶ $\forall x \in Z_n^*$ $x^{2m} \equiv 1 \pmod n$ implies that $\forall y \in QR_n$ $ord_n(y) \mid p'q'$ i.e. $ord_n(y)$ is either 1, p', q', or p'q' (if there is one y s.t. $ord_n(y)=m$ then y is a generator and QR_n is cyclic). Let's construct one.

Subgroups in Z_n^* (cont'd)

Let g_1 be a generator in Z_p^*, and g_2 be a generator in Z_q^*

Let $\mathbf{g \equiv g_1 \pmod p \equiv g_2 \pmod q}$, (note that J(g, n) = 1, $g \in J_{11}$)

$g^{p-1} \equiv g^{2p'} \equiv g_1^{2p'} \equiv 1 \pmod p$, $g^{q-1} \equiv g^{2q'} \equiv g_2^{2q'} \equiv 1 \pmod q$

$\Rightarrow$ $g^{2p'q'} \equiv 1 \pmod p$ and $g^{2q'p'} \equiv 1 \pmod q$ i.e. $g^{2p'q'} \equiv 1 \pmod n$

if there exists a $k \in \{1, 2, p', q', 2p', 2q', p'q'\}$ s.t. $g^k \equiv 1 \pmod n$

then $ord_n(g)$ is not 2p'q'

1. k=1: $\Rightarrow g_1 \equiv 1 \pmod p$ contradict with $ord_p(g_1) = p-1$
2. k=p': $\Rightarrow g^{p'} \equiv g_1^{p'} \equiv 1 \pmod p$ contradict with $ord_p(g_1) = 2p'$
3. k=q': $\Rightarrow g^{q'} \equiv g_2^{q'} \equiv 1 \pmod q$ contradict with $ord_q(g_2) = 2q'$
4. k=2: $\Rightarrow g_1^2 \equiv 1 \pmod p$ contradict with $ord_p(g_1) = p-1$
5. k=2p': $\Rightarrow g^{2p'} \equiv g_2^{2p'} \equiv 1 \pmod q$ contradict with $ord_q(g_2) = 2q'$
6. k=2q': $\Rightarrow g^{2q'} \equiv g_1^{2q'} \equiv 1 \pmod p$ contradict with $ord_p(g_1) = 2p'$

Subgroups in Z_n^* (cont'd)

7. k=p'q': $\Rightarrow g^{p'q'} \equiv g_1^{p'q'} \equiv 1 \pmod p$

since $g_1^{2p'} \equiv 1 \pmod p$ and

$\gcd(q', 2) = 1 \Rightarrow \exists$ a, b s.t. a q' + b 2 = 1

$\Rightarrow g_1^{p'} \equiv g_1^{p'(a q' + b 2)} \equiv (g_1^{p' q'})^a (g_1^{2 p'})^b \equiv 1 \pmod p$

contradict with $ord_p(g_1) = 2p'$

1~7 implies that $ord_n(g) = 2p'q'$, i.e. $QR_o = \{g^2, g^4, \ldots, g^{p'q'}\}$ and QR_n is a cyclic group.

- ✶ Pr{Elements in QR_n being a generator} = $\phi(p'q')$ / (p'q')

- ✧ $\mathbf{J_n}$ is a cyclic subgroup of order $2m = \lambda(n)$, $J_n = \{x \in Z_n^* \mid J(x,n)=1\}$
  - ✶ $J_{11} = \{x \in Z_n^* \mid J(x,p)=-1 \text{ and } J(x,q)=-1\}$
  - ✶ The above proof also shows that $J_n = \{g, g^2, \ldots, g^{2p'q'}\}$ is cyclic
  - ✶ Pr{Elements in J_n being a generator} = $\phi(p'q')$ / (2p'q')
- ✧ $\mathbf{J_{01} \cup J_{10}} = Z_n^* \setminus \{J_{00} \cup J_{11}\}$ is not a subgroup in Z_n^*
  - ✶ if $x \in J_{01}$ then $x * x \in J_{00}$

Generator in QR_n

- ✧ n = p q, p=2p'+1, q=2q'+1
- ✧ Find a generator in QR_n

1. Find a generator g_1 of Z_p^* (i.e. $Z_p^* = <g_1>$) and g_2 of Z_q^* (i.e. $Z_q^* = <g_2>$)
2. Calculate the generator $h_1 \equiv g_1^2 \pmod p$ of QR_p and $h_2 \equiv g_2^2$ (mod 1) of QR_q
3. Let $h \equiv h_1 \pmod p \equiv h_2 \pmod q$.

   It is clear that $h \equiv g^2 \pmod n$, i.e. $h \in QR_n$, where $g \equiv g_1 \pmod p \equiv g_2 \pmod q$.

Claim: h is a generator of QR_n

pf.

$y \in QR_n \Rightarrow$ $y \in QR_p$ and $y \in QR_q$

i.e. $\exists$ $x_1 \in Z_{p'}$ and $x_2 \in Z_{q'}$, $y \equiv h_1^{x_1} \pmod p \equiv h_2^{x_2} \pmod q$

$\Rightarrow y \equiv g_1^{2 x_1} \pmod p \equiv g_2^{2 x_2} \pmod q$

$\Rightarrow y \equiv g^{2x} \pmod n$ if $2x \equiv 2 x_1 \pmod{p-1} \equiv 2 x_2 \pmod{q-1}$

a unique $x \in Z_{p'q'}$ exists by CRT since gcd(p-1, q-1) = gcd(2p', 2q') = 2

$\Rightarrow y \equiv h^x \pmod n$

Generate Elements in Z_n^*

- Z_n^* is NOT a cyclic group (n = p q, p=2p'+1, q=2q'+1, m=p' q')
- How do we generate random elements in Z_n^*?

  $Z_n^* = \{ g^a u^{-e\,b_1} (-1)^{b_2} \mid g$ is a generator in QR_n, $\gcd(e, \phi(n)) = 1$,
  $u \in_R Z_n^*$ and $J(u,n) = -1$,
  $a \in \{0,\ldots,m-1\}$, $b_1 \in \{0,1\}$, and $b_2 \in \{0,1\} \}$

  Note: 1. $J(-1, n) = 1$ and $-1 \in J_n \backslash QR_n$ since $(-1)^{(p-1)/2} \equiv (-1)^{p'} \equiv -1 \pmod p$
  2. e is odd, $\phi(n)$-e is also odd, $J(u^{-e}, n) = J(u, n) = -1$
- We can view the above as 4 parts
  1. J_{00} (QR_n): $b_1 = b_2 = 0$, $J_{00} = \{g^a \mid a \in \{0,\ldots,m-1\}\}$
  2. J_{11} ($J_n \backslash QR_n$): $b_1 = 0$, $b_2 = 1$, $J_{11} = \{-g^a \mid a \in \{0,\ldots,m-1\}\}$

  Assume that $J(u, p) = -1$ and $J(u, q) = 1$

  3. J_{01}: $b_1 = 1$, $b_2 = 0$, $J_{01} = \{g^a u^{-e} \mid a \in \{0,\ldots,m-1\}\}$
  4. J_{10}: $b_1 = 1$, $b_2 = 1$, $J_{01} = \{-g^a u^{-e} \mid a \in \{0,\ldots,m-1\}\}$

Lagrange's Theorem

- Theorem: for any finite group G, the order (number of elements) of every subgroup H of G divides the order of G.
  - proof sketch: divide G into left cosets H – equivalence classes, and show that they have the same size.
- It implies that: the order of any element *a* of a finite group (i.e. the smallest positive integer number *k* with $a^k = 1$) divides the order of the group. Since the order of *a* is equal to the order of the cyclic subgroup generated by *a*. Also, $a^{|G|} = 1$ since order of *a* divides |G|.
- Any prime order group is cyclic.