Number Theory for Cryptography

密碼學與應用
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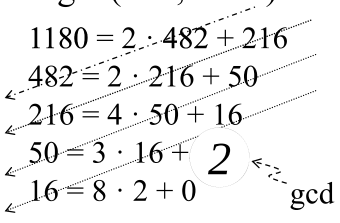
Congruence

- **Modulo Operation:**
  - **Question:** What is 12 mod 9?
  - **Answer:** 12 mod 9 ≡ 3 or 12 ≡ 3 (mod 9)
    "12 is congruent to 3 modulo 9"
- **Definition:** Let $a, r, m \in Z$ (where Z is the set of all integers) and $m > 0$. We write
  - $a \equiv r \pmod{m}$ if m divides $a - r$ (i.e. $m \mid a-r$)
  - m is called the *modulus*
  - r is called the *remainder*
  - $a = q \cdot m + r \qquad 0 \le r < m$
- **Example:** $a = 42$ and $m=9$
  - $42 = 4 \cdot 9 + 6$ therefore $42 \equiv 6 \pmod{9}$

Greatest Common Divisor

- GCD of a and b is the largest positive integer dividing both a and b
- gcd(a, b) or (a,b)
- ex. gcd(6, 4) = 2, gcd(5, 7) = 1
- Euclidean algorithm remainder→divisor → dividend → ignore
  - ex. gcd(482, 1180)



$1180 = 2 \cdot 482 + 216$

$482 = 2 \cdot 216 + 50$

$216 = 4 \cdot 50 + 16$

$50 = 3 \cdot 16 + 2$

$16 = 8 \cdot 2 + 0$

Why does it work?

Let d = gcd(482, 1180)
$d \mid 482$ and $d \mid 1180 \Rightarrow d \mid 216$
because $216 = 1180 - 2 \cdot 482$
$d \mid 216$ and $d \mid 482 \Rightarrow d \mid 50$
$d \mid 50$ and $d \mid 216 \Rightarrow d \mid 16$
$d \mid 16$ and $d \mid 50 \Rightarrow d \mid 2$
$2 \mid 16 \Rightarrow d = 2$

Greatest Common Divisor (cont'd)

- Euclidean Algorithm: calculating GCD

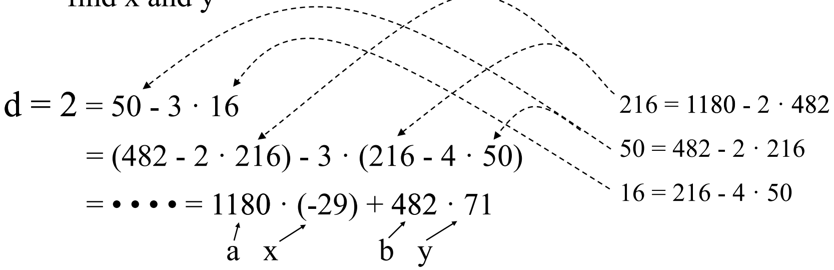
gcd(1180, 482)

(輾轉相除法)

2	482 432	1180 964	2
3	50 48	216 200	4
	2	16 16	8
		0	

Greatest Common Divisor (cont'd)

- ✧ Def: a and b are relatively prime: gcd(a, b) = 1
- ✧ Theorem: Let a and b be two integers, with at least one of a, b nonzero, and let d = gcd(a,b). Then there exist integers x, y, gcd(x, y) = 1 such that a · x + b · y = d
  - ✶ Constructive proof: Using Extended Euclidean Algorithm to find x and y



$$d = 2 = 50 - 3 \cdot 16$$

$$= (482 - 2 \cdot 216) - 3 \cdot (216 - 4 \cdot 50)$$

$$= \bullet\bullet\bullet\bullet = \underset{a}{1180} \cdot \underset{x}{(-29)} + \underset{b}{482} \cdot \underset{y}{71}$$

216 = 1180 - 2 · 482

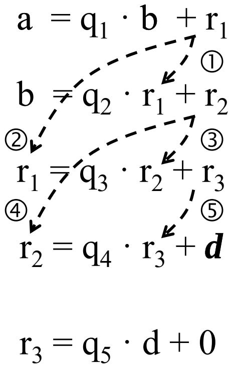
50 = 482 - 2 · 216

16 = 216 - 4 · 50

Extended Euclidean Algorithm

Let gcd(a, b) = d

- ✧ Looking for s and t, gcd(s, t) = 1 s.t. a · s + b · t = d
- ✧ When d = 1, t ≡ b^{-1} (mod a)



$a = q_1 \cdot b + r_1$

①

$b = q_2 \cdot r_1 + r_2$

② ③

$r_1 = q_3 \cdot r_2 + r_3$

④ ⑤

$r_2 = q_4 \cdot r_3 + \boldsymbol{d}$

$r_3 = q_5 \cdot d + 0$

Ex. 1180 = 2 · 482 + 216
1180 - 2 · 482 = 216
482 = 2 · 216 + 50
482 - 2 · (1180 - 2 · 482) = 50
-2 · 1180 + 5 · 482 = 50
216 = 4 · 50 + 16
(1180 - 2 · 482) -
4 · (-2 · 1180 + 5 · 482) = 16
9 · 1180 - 22 · 482 = 16
50 = 3 · 16 + ***2***
(-2 · 1180 + 5 · 482) -
3 · (9 · 1180 - 22 · 482) = 2
-29 · 1180 + 71 · 482 = 2

Greatest Common Divisor (cont'd)

- ✶ The above proves only the existence of integers x and y
- ✶ How about gcd(x, y)?

∈Z

$$\left.\begin{array}{l} d = a \cdot x + b \cdot y \\ d = \gcd(a, b) \end{array}\right. \Rightarrow \quad 1 = a/d \cdot x + b/d \cdot y$$

If gcd(x, y) = r , r ≥ 1 then

r | x and r | y ⇒ r | a/d · x + b/d · y

which means that r | 1 i.e. r = 1

gcd(x, y) = 1 ¶

Note: gcd(x, y) = 1 but (x, y) is not unique

e.g. d = a x + b y = a (x-k·b) + b (y+k·a)

when k increases, x-k·b decreases and become negative

Greatest Common Divisor (cont'd)

Lemma: gcd(a,b) = gcd(x,y) = gcd(a,y) = gcd(x,b) = 1 ⇔

∃ a, b, x, y s.t. 1 = a x + b y

pf:

(⇒) following the previous theorem

(⇐) let d = gcd(a, b), d ≥ 1

⇒ d | a and d | b

⇒ d | a x + b y = 1

⇒ d = 1

similarly, gcd(a, y)=1, gcd(x, b)=1, and gcd(x, y)=1

Operations under mod n

✧ Proposition:

Let a,b,c,d,n be integers with $n \neq 0$, suppose

$a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$ then

$a + c \equiv b + d \pmod{n}$

$a - c \equiv b - d \pmod{n}$

$a \cdot c \equiv b \cdot d \pmod{n}$

pf. $\begin{cases} a = k_1 n + b \\ c = k_2 n + d \end{cases}$

$\Rightarrow (a+c) = (k_1+k_2)\, n + (b+d)$

$\Rightarrow a+c \equiv b+d \pmod{n}$

✧ Proposition:

Let a,b,c,n be integers with $n \neq 0$ and $\gcd(a,n) = 1$.

If $a \cdot b \equiv a \cdot c \pmod{n}$ then $b \equiv c \pmod{n}$

Operations under mod n

✧ What is the multiplicative inverse of a (mod n)?

i.e. $a \cdot a^{-1} \equiv 1 \pmod{n}$ or $a \cdot a^{-1} = 1 + k \cdot n$

$\gcd(a, n) = 1 \Rightarrow \exists$ s and t such that $a \cdot s + n \cdot t = 1$

Extended Euclidean Algo.

$\Rightarrow a^{-1} \equiv s \pmod{n}$

This expression als
implies gcd(a,n)=

✧ $a \cdot x \equiv b \pmod{n}$, $\gcd(a, n) = 1$, $x \equiv ?$

$x \equiv b \cdot a^{-1} \equiv b \cdot s \pmod{n}$

Are there any solutio

✧ $a \cdot x \equiv b \pmod{n}$, $\gcd(a, n) = d > 1$, $x \equiv ?$

if $d \mid b$ $(a/d) \cdot x \equiv (b/d) \pmod{n/d}$ $\gcd(a/d, n/d) = 1$

$x_0 \equiv (b/d) \cdot (a/d)^{-1} \pmod{n/d}$

$\Rightarrow$ there are d solutions to the equation $a \cdot x \equiv b \pmod{n}$:

$x_0, x_0+(n/d), \ldots, x_0+(d-1)\cdot(n/d) \pmod{n}$

Matrix inversion under mod n

✧ A square matrix is invertible mod n if and only if its determinant and n are relatively prime

✧ ex: in real field R

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

In a finite field Z (mod n)? we need to find the inverse for ad-bc (mod n) in order to calculate the inverse of the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \equiv (ad - bc)^{-1} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \pmod{n}$$

Group

✧ A group G is a finite or infinite set of elements and a binary operation × which together satisfy

1. Closure: $\forall\, a,b \in G$ $a \times b = c \in G$ 封閉性
2. Associativity: $\forall\, a,b,c \in G$ $(a \times b) \times c = a \times (b \times c)$ 結合性
3. Identity: $\forall\, a \in G$ $1 \times a = a \times 1 = a$ 單位元素
4. Inverse: $\forall\, a \in G$ $a \times a^{-1} = 1 = a^{-1} \times a$ 反元素

✧ Abelian group 交換群 $\forall\, a,b \in G$ $a \times b = b \times a$

means $g \times g \times g \times \ldots \times g$

✧ Cyclic group G of order m: a group defined by an element $g \in G$ such that $g, g^2, g^3, \ldots.\, g^m$ are all distinct elements in G (thus cover all elements of G) and $g^m = 1$, the element g is called a generator of G. Ex: Z_n^* (or Z/nZ)

Group (cont'd)

- The **order of a group**: the number of elements in a group G, denoted |G|. If the order of a group is a finite number, the group is said to be a finite group, note $g^{|G|} = 1$ (the identity element).
- The **order of an element g** of a finite group G is the smallest power m such that $g^m = 1$ (the identity element), denoted by $\mathrm{ord}_G(g)$
- ex: $\mathbf{Z_n}$: additive group modulo n is the set {0, 1, …, n-1}
  binary operation: + (mod n)
  identity: 0
  inverse: $-x \equiv n-x \pmod n$ Algorithm

  size of Z_n is n,
  $\underbrace{g+g+\ldots+g} \equiv 0 \pmod n$
- ex: $\mathbf{Z_n^*}$: multiplicative group modulo n is the set {i:0<i<n, gcd(i,n)=1}
  binary operation: × (mod n)
  identity: 1
  inverse: x^{-1} can be found using extended Euclidean Algorithm

  size of Z_n^* is $\phi(n)$,
  $g^{\phi(n)} \equiv 1 \pmod n$

Ring Z_m

- **Definition:** The ring Z_m consists of
  - The set $Z_m = \{0, 1, 2, \ldots, m-1\}$
  - Two operations "+ (mod m)" and "× (mod m)" for all $a, b \in Z_m$ such that they satisfy the properties on the next slide
- **Example:** $m = 9$ $Z_9 = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$
  $6 + 8 = 14 \equiv 5 \pmod 9$
  $6 \times 8 = 48 \equiv 3 \pmod 9$

Properties of the ring Z_m

- Consider the ring $Z_m = \{0, 1, \ldots, m-1\}$
  - The additive identity "0": $a + 0 \equiv a \pmod m$
  - The additive inverse of a: $-a = m - a$ s.t. $a + (-a) \equiv 0 \pmod m$
  - Addition is closed i.e if $a, b \in Z_m$ then $a + b \in Z_m$
  - Addition is associative $(a + b) + c \equiv a + (b + c) \pmod m$
  - Addition is commutative $a + b \equiv b + a \pmod m$
  - Multiplicative identity "1": $a \times 1 \equiv a \pmod m$
  - The multiplicative inverse of a exists only when $\gcd(a,m) = 1$ and denoted as a^{-1} s.t. $a^{-1} \times a \equiv 1 \pmod m$ **might or might not exist**
  - Multiplication is closed i.e. if $a, b \in Z_m$ then $a \times b \in Z_m$
  - Multiplication is associative $(a \times b) \times c \equiv a \times (b \times c) \pmod m$
  - Multiplication is commutative $a \times b \equiv b \times a \pmod m$

Some remarks on the ring Z_m

- A ring is an Abelian group under addition and an Abelian semigroup under multiplication..
- A semigroup is defined for a **set** and an associative **binary operator**. No other restrictions are placed on a semigroup; thus a semigroup need not have an **identity** element and its elements need not have **inverses** within the semigroup.

Some remarks on the ring Z_m (cont'd)

- Roughly speaking a ring is a mathematical structure in which we can add, subtract, multiply, and even sometimes divide. (A ring in which every element has multiplicative inverse is called a field.)
  - **Example:** Is the division 4/15 (mod 26) possible?

    In fact, $4/15 \bmod 26 \equiv 4 \times 15^{-1} \pmod{26}$

    Does $15^{-1} \pmod{26}$ exist ?

    It exists only if gcd(15, 26) = 1.

    $15^{-1} \equiv 7 \pmod{26}$ therefore,

    $4/15 \bmod 26 \equiv 4 \times 7 \equiv 28 \equiv 2 \bmod 26$

Some remarks on the group Z_m *and* Z_m^*

- The modulo operation can be applied whenever we want

in Z_m

$$(a + b) \pmod{m} \equiv [(a \pmod{m}) + ((b \bmod m))] \pmod{m}$$

in Z_m^*

$$(a \times b) \pmod{m} \equiv [(a \pmod{m}) \times ((b \bmod m))] \pmod{m}$$

$$a^b \pmod{m} \equiv (a \pmod{m})^b \pmod{m}$$

👓 Question? $a^b \pmod{m} \overset{?}{\equiv} a^{\boxed{(b \bmod m)}} \pmod{m}$

Exponentiation in Z_m

- Example: $3^8 \pmod{7} \equiv ?$

  $3^8 \pmod{7} \equiv 6561 \pmod{7} \equiv 2$ since $6561 \equiv 937 \times 7 + 2$

  or

  $3^8 \pmod{7} \equiv 3^4 \times 3^4 \pmod{7} \equiv 3^2 \times 3^2 \times 3^2 \times 3^2 \pmod{7}$

  $\equiv (3^2 \pmod{7})\times(3^2 \pmod{7})\times(3^2 \pmod{7})\times(3^2 \pmod{7})$

  $\equiv 2 \times 2 \times 2 \times 2 \pmod{7} \equiv 16 \pmod{7} \equiv 2$

- The cyclic group Z_m^* and the modulo arithmetic is of central importance to modern public-key cryptography. In practice, the order of the integers involved in PKC are in the range of $[2^{160}, 2^{1024}]$. Perhaps even larger.

Exponentiation in Z_m (cont'd)

- How do we do the exponentiation efficiently?
- $3^{1234} \pmod{789}$ many ways to do this

  a. do 1234 times multiplication and then calculate remainder

  b. repeat 1234 times (multiplication by 3 and calculate remainder)

  c. repeated $\lfloor \log 1234 \rfloor$ times (square, multiply and calculate remainder)

  ex. first tabulate

  $3^2 \equiv 9 \pmod{789}$

  $3^4 \equiv 9^2 \equiv 81$

  $3^8 \equiv 81^2 \equiv 249$

  $3^{16} \equiv 249^2 \equiv 459$

  $3^{32} \equiv 459^2 \equiv 18$

  $3^{64} \equiv 18^2 \equiv 324$

  $3^{128} \equiv 324^2 \equiv 39$

  $3^{256} \equiv 39^2 \equiv 732$

  $3^{512} \equiv 732^2 \equiv 93$

  $3^{1024} \equiv 93^2 \equiv 759$

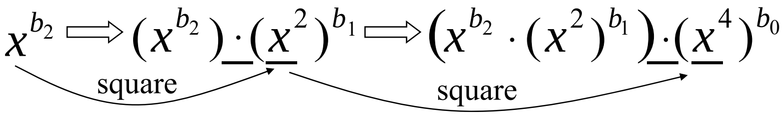
  $1234 = 1024 + 128 + 64 + 16 + 2$ $(10011010010)_2$

  $3^{1234} \equiv 3^{(1024+128+64+16+2)} \equiv (((759 \bullet 39) \bullet 324) \bullet 459) \bullet 9 \equiv 105 \pmod{789}$

Exponentiation in Z_m (cont'd)

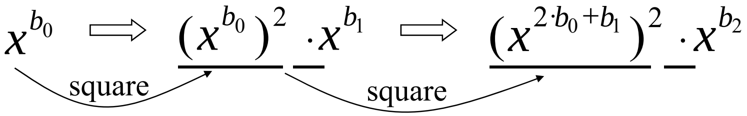
calculate x^y (mod m) where $y = b_0 \cdot 2^2 + b_1 \cdot 2 + b_2$

- Method 1:



$$x^{b_2} \Rightarrow (x^{b_2})\underline{\cdot}(\underline{x}^2)^{b_1} \Rightarrow \left(x^{b_2}\cdot(x^2)^{b_1}\right)\underline{\cdot}(\underline{x}^4)^{b_0}$$

- Method 2:



$$x^{b_0} \Rightarrow \underline{(x^{b_0})^2}\ \underline{\cdot}\, x^{b_1} \Rightarrow \underline{(x^{2\cdot b_0+b_1})^2}\ \underline{\cdot}\, x^{b_2}$$

square and **multiply** $\lfloor \log y \rfloor$ times

Exponentiation in Z_m (cont'd)

Method 1:

$1234 = 1024 + 128 + 64 + 16 + 2 \qquad (10011010010)_2$

$3^{1234} \equiv 3^{0+2(1+2(0+2(0+2(1+2(0+2(1+2(1+2(0+2(0+2(1))))))))))}$

$\equiv 9 \bullet 9^{2(0+2(0+2(1+2(0+2(1+2(1+2(0+2(0+2(1)))))))))}$

$\equiv 9 \bullet 81^{2(0+2(1+2(0+2(1+2(1+2(0+2(0+2(1))))))))}$

$\equiv 9 \bullet 249^{2(1+2(0+2(1+2(1+2(0+2(0+2(1)))))))}$

$\equiv 9 \bullet 459 \bullet 459^{2(0+2(1+2(1+2(0+2(0+2(1))))))}$

$\equiv 9 \bullet 459 \bullet 18^{2(1+2(1+2(0+2(0+2(1)))))}$

$\equiv 9 \bullet 459 \bullet 324 \bullet 324^{2(1+2(0+2(0+2(1))))}$

$\equiv 9 \bullet 459 \bullet 324 \bullet 39 \bullet 39^{2(0+2(0+2(1)))}$

$\equiv 9 \bullet 459 \bullet 324 \bullet 39 \bullet 732^{2(0+2(1))}$

$\equiv 9 \bullet 459 \bullet 324 \bullet 39 \bullet 93^{2\,(1)}$

$\equiv 9 \bullet 459 \bullet 324 \bullet 39 \bullet 759 \bmod 789$

Exponentiation in Z_m (cont'd)

Method 2: $1234 = 1024 + 128 + 64 + 16 + 2 \qquad (10011010010)_2$

$3^{1234} \equiv 3^{0+2(1+2(0+2(0+2(1+2(0+2(1+2(1+2(0+2(0+2(1))))))))))}$

$\equiv (3\bullet 3^{2(0+2(1+2(0+2(1+2(1+2(0+2(0+2(1))))))))})^2$

$\equiv (3\bullet(3^{2(1+2(0+2(1+2(1+2(0+2(0+2(1)))))))})^2)^2$

$\equiv (3\bullet((3\bullet 3^{2(0+2(1+2(1+2(0+2(0+2(1))))))})^2)^2)^2$

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$\equiv (3\bullet((3\bullet((3\bullet(3\bullet(3^{2(0+2(1))})^2)^2)^2)^2)^2)^2)^2$

$\equiv (3\bullet((3\bullet((3\bullet(3\bullet((3^{2(1)})^2)^2)^2)^2)^2)^2)^2)^2$

$\equiv (3\bullet((3\bullet((3\bullet(3\bullet((\underline{(3^1)}^2)^2)^2)^2)^2)^2)^2)^2)^2$

Chinese Remainder Theorem (CRT)

- $\forall\ i \neq j \in \{1,2,\ldots k\}$, $\gcd(r_i, r_j) = 1$, $0 \le m_i < r_i$
  Is there an m that satisfies simultaneously the following set of congruence equations?

$$\begin{aligned} m &\equiv m_1 \pmod{r_1} \\ &\equiv m_2 \pmod{r_2} \\ &\ \bullet\bullet\bullet \\ &\equiv m_k \pmod{r_k} \end{aligned}$$

ex: $m \equiv 1 \pmod 3$
$\equiv 2 \pmod 5$
$\equiv 3 \pmod 7$
Note: $\gcd(3,5) = 1$
$\gcd(3,7) = 1$
$\gcd(5,7) = 1$

- 韓信點兵: 三個一數餘一, 五個一數餘二, 七個一數餘三, 請問隊伍中至少有幾名士兵?

Chinese Remainder Theorem (CRT)

✧ first solution:

$n = r_1\, r_2 \cdot\cdot\cdot\, r_k$

$z_i = n / r_i$

$\exists!\ s_i \in Z^*_{r_i}$ s.t. $s_i \cdot z_i \equiv 1 \pmod{r_i}$ (since $\gcd(z_i, r_i) = 1$)

$$m \equiv \sum_{i=1}^{k} z_i \cdot s_i \cdot m_i \pmod{n}$$

Unique solution in Z_n?

✧ ex: $m_1=1,\ m_2=2,\ m_3=3$

$r_1=3,\quad r_2=5,\quad r_3=7 \qquad n = 3 \cdot 5 \cdot 7$

$z_1=35,\ z_2=21,\ z_3=15$

$s_1=2,\quad s_2=1,\quad s_3=1 \qquad 35 \cdot 2 + 3\,(-23) = 1$

$m \equiv 35\cdot 2\cdot 1 + 21\cdot 1\cdot 2 + 15\cdot 1\cdot 3 \equiv 157 \equiv 52 \pmod{105}$

Chinese Remainder Theorem (CRT)

✧ Uniqueness:

1. If there exists $m' \in Z_n$ ($\neq m$) also satisfies the previous k congruence relations, then
   $\forall i,\ m'-m \equiv 0 \pmod{r_i}$.
2. This is equivalent to $\forall i,\ r_i \mid m' - m$
3. $\forall i,j,\ \gcd(r_i, r_j) = 1 \Rightarrow r_1\, r_2 \ldots r_k \mid m' - m$

⇨ $m' = m + k \cdot r_1, r_2 \ldots r_k = m + k \cdot n$

⇨ $m' \notin Z_n$ for all $k \neq 0$

contradiction!

Chinese Remainder Theorem (CRT)

✧ second solution:

$R_i = r_1\, r_2 \cdot\cdot\cdot\, r_{i-1}$

$\exists!\ t_i \in Z^*_{r_i}$ s.t. $t_i \cdot R_i \equiv 1 \pmod{r_i}$ (since $\gcd(R_i, r_i) = 1$)

$$\begin{cases} \hat{m}_1 = m_1 \\ \hat{m}_i = \hat{m}_{i-1} + R_i \cdot (m_i - \hat{m}_{i-1}) \cdot t_i \pmod{R_{i+1}} & i \geq 2 \\ m = \hat{m}_k \end{cases}$$

satisfies the first i-1 congruence relations

Note that $\hat{m}_i \equiv m_1 \pmod{r_1}$

$\equiv m_2 \pmod{r_2}$

• • •

$\equiv m_i \pmod{r_i}$

$m_1=1,\ m_2=2,\ m_3=3$

$r_1=3,\quad r_2=5,\quad r_3=7$

$R_2=3,\ R_3=15,\ R_4=105$

$t_2=2,\quad t_3=1$

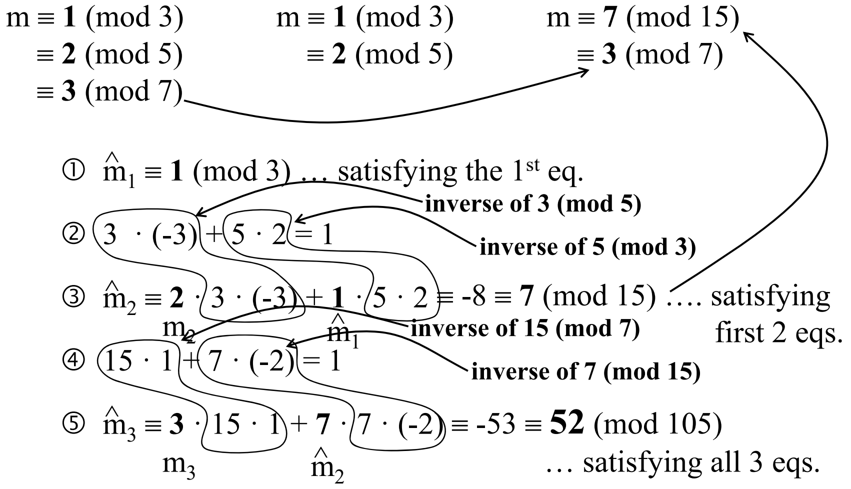
ex: $\hat{m}_1 \equiv 1$

$\hat{m}_2 \equiv 1+3\cdot(2-1)\cdot 2=7$

$\hat{m} \equiv m_3 \equiv 7+15\cdot(3-7)\cdot 1$

$\equiv -53 \equiv 52 \pmod{105}$

Incremental Manual Calculation



$m \equiv \mathbf{1} \pmod{3}$ $\qquad m \equiv \mathbf{1} \pmod{3}$ $\qquad m \equiv \mathbf{7} \pmod{15}$

$\equiv \mathbf{2} \pmod{5}$ $\qquad \equiv \mathbf{2} \pmod{5}$ $\qquad \equiv \mathbf{3} \pmod{7}$

$\equiv \mathbf{3} \pmod{7}$

① $\hat{m}_1 \equiv \mathbf{1} \pmod{3}$ … satisfying the 1st eq.

② $3 \cdot (-3) + 5 \cdot 2 = 1$

③ $\hat{m}_2 \equiv \mathbf{2} \cdot 3 \cdot (-3) + \mathbf{1} \cdot 5 \cdot 2 \equiv -8 \equiv \mathbf{7} \pmod{15}$ …. satisfying first 2 eqs.

④ $15 \cdot 1 + 7 \cdot (-2) = 1$

⑤ $\hat{m}_3 \equiv \mathbf{3} \cdot 15 \cdot 1 + \mathbf{7} \cdot 7 \cdot (-2) \equiv -53 \equiv \mathbf{52} \pmod{105}$

… satisfying all 3 eqs.

Chinese Remainder Theorem (CRT)

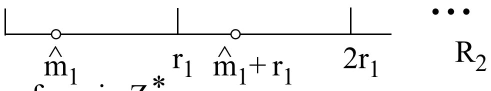
✧ special case:

$x \equiv m \pmod{r_1} \equiv m \pmod{r_2} \bullet\bullet\bullet \equiv m_n \pmod{r_n} \Rightarrow x \equiv m \pmod{r_1 r_2 \bullet\bullet\bullet r_n}$

✧ insight of the second solution: **every step satisfies one more equation**

step 1

$x \equiv m_1 \pmod{r_1}$



$R_2 = r_1$

let $\hat{m}_1 = m_1$

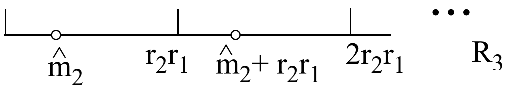
m_1 is the only solution for x in $Z^*_{R_2}$

general solution of x must be $\hat{m}_1 + k\, R_2$ for some k

step 2

$x \equiv m_1 \pmod{r_1}$

$\equiv m_2 \pmod{r_2}$



$R_3 = r_2 r_1$

let $\hat{m}_2 \equiv \hat{m}_1 + k^* R_2 \pmod{R_3}$ where $k^* = t_2(m_2 - \hat{m}_1)$ and $t_2 R_2 \equiv 1 \pmod{r_2}$

m_2 is the only solution for x in $Z^*_{R_3}$

general solution of x must be $\hat{m}_2 + k\, R_3$ for some k

Chinese Remainder Theorem (CRT)

✧ Applications: solve $x^2 \equiv 1 \pmod{35}$

- ✶ $35 = 5 \cdot 7$
- ✶ x* satisfies $f(x^*) \equiv 0 \pmod{35}$ ⇔ x* satisfies both $f(x^*) \equiv 0 \pmod 5$ and $f(x^*) \equiv 0 \pmod 7$

Proof:

(⇐)

$p \mid f(x^*)$, $q \mid f(x^*)$, and $\gcd(p,q)=1$ imply that
$p \cdot q \mid f(x^*)$ i.e. $f(x^*) \equiv 0 \pmod{p \cdot q}$

(⇒)

$f(x^*) = k \cdot p \cdot q$ implies that
$f(x^*) = (k \cdot p) \cdot q = (k \cdot q) \cdot p$ i.e. $f(x^*) \equiv 0 \pmod p$
$\equiv 0 \pmod q$

Chinese Remainder Theorem (CRT)

- ✶ since 5 and 7 are prime, we can solve
  $x^2 \equiv 1 \pmod 5$ and $x^2 \equiv 1 \pmod 7$
  far more easily than $x^2 \equiv 1 \pmod{35}$

Why?

  - ✡ $x^2 \equiv 1 \pmod 5$ has exactly two solutions: $x \equiv \pm 1 \pmod 5$
  - ✡ $x^2 \equiv 1 \pmod 7$ has exactly two solutions: $x \equiv \pm 1 \pmod 7$
- ✶ put them together and use CRT, there are four solutions
  - ✡ $x \equiv 1 \pmod 5 \equiv 1 \pmod 7 \Rightarrow x \equiv 1 \pmod{35}$
  - ✡ $x \equiv 1 \pmod 5 \equiv 6 \pmod 7 \Rightarrow x \equiv 6 \pmod{35}$
  - ✡ $x \equiv 4 \pmod 5 \equiv 1 \pmod 7 \Rightarrow x \equiv 29 \pmod{35}$
  - ✡ $x \equiv 4 \pmod 5 \equiv 6 \pmod 7 \Rightarrow x \equiv 34 \pmod{35}$

Matlab tools

	format rat format long
matrix inverse	inv(A)
matrix determinant	det(A)
p = q d + r	r = mod(p, d) or r = rem(p, d)
	q = floor(p / d)
	g = gcd(a, b)
g = a s + b t	[g, s, t] = gcd(a, b)
factoring	factor(N)
prime numbers < N	primes(N)
test prime	isprime(p)
mod exponentiation *	powermod(a,b,n)
find primitive root *	primitiveroot(p)
crt *	crt([a_1 a_2 a_3...], [m_1 m_2 m_3...])
ϕ(N) *	eulerphi(N)

Field

- Field: a set that has the operation of addition, multiplication, subtraction, and division by nonzero elements. Also, the associative, commutative, and distributive laws hold.
- Ex. Real numbers, complex numbers, rational numbers, integers mod a prime are fields
- Ex. Integers, 2×2 matrices with real entries are not fields
- Ex. GF(4) = $\{0, 1, \omega, \omega^2\}$
  - $0 + x = x$
  - $x + x = 0$
  - $1 \cdot x = x$
  - $\omega + 1 = \omega^2$

- Addition and multiplication are commutative and associative, and the distributive law $x(y+z)=xy+xz$ holds for all x, y, z
- $x^3 = 1$ for all nonzero elements

Galois Field

- Galois Field: A field with finite element, finite field
- For every power p^n of a prime, there is exactly one finite field with p^n elements, GF(p^n), and these are the only finite fields.
- For n > 1, {integers (mod p^n)} do not form a field.
  - Ex. $p \cdot x \equiv 1 \pmod{p^n}$ does not have a solution (i.e. p does not have multiplicative inverse)

How to construct a GF(p^n)?

- Def: $Z_2[X]$: the set of polynomials whose coefficients are integers mod 2
  - ex. 0, 1, $1+X^3+X^6\ldots$
  - add/subtract/multiply/divide/Euclidean Algorithm: process all coefficients mod 2
    - $(1+X^2+X^4) + (X+X^2) = 1+X+X^4$ bitwise XOR
    - $(1+X+X^3)(1+X) = 1+X^2+X^3+X^4$
    - $X^4+X^3+1 = (X^2+1)(X^2+X+1) + X$ long division can be written as $X^4+X^3+1 \equiv X \pmod{X^2+X+1}$

How to construct GF(2^n)?

- Define $Z_2[X] \pmod{X^2+X+1}$ to be $\{0, 1, X, X+1\}$
  - addition, subtraction, multiplication are done mod X^2+X+1
  - $f(X) \equiv g(X) \pmod{X^2+X+1}$
    - if f(X) and g(X) have the same remainder when divided by X^2+X+1
    - or equivalently $\exists$ h(X) such that $f(X) - g(X) = (X^2+X+1)\, h(X)$
    - ex. $X \cdot X = X^2 \equiv X+1 \pmod{X^2+X+1}$
  - if we replace X by ω, we can get the same GF(4) as before
  - the modulus polynomial X^2+X+1 should be irreducible

Irreducible: polynomial does not factor into polynomials of lower degree with mod 2 arithmetic
ex. X^2+1 is not irreducible since $X^2+1 = (X+1)(X+1)$

How to construct GF(p^n)?

- $Z_p[X]$ is the set of polynomials with coefficients mod p
- Choose P(X) to be any one irreducible polynomial mod p of degree n (other irreducible P(X)'s would result to isomorphisms)
- Let GF(p^n) be $Z_p[X]$ mod P(X)

- An element in $Z_p[X]$ mod P(X) must be of the form
  $$a_0 + a_1 X + \ldots + a_{n-1} X^{n-1}$$
  each a_i are integers mod p, and have p choices, hence there are p^n possible elements in GF(p^n)
- multiplicative inverse of any element in GF(p^n) can be found using extended Euclidean algorithm (over polynomial)

GF(2^8)

- AES (Rijndael) uses GF(2^8) with irreducible polynomial $X^8 + X^4 + X^3 + X + 1$
- each element is represented as
  $$b_7 X^7 + b_6 X^6 + b_5 X^5 + b_4 X^4 + b_3 X^3 + b_2 X^2 + b_1 X + b_0$$
  each b_i is either 0 or 1
- elements of GF(2^8) can be represented as 8-bit bytes $b_7b_6b_5b_4b_3b_2b_1b_0$
- mod 2 operations can be implemented by XOR in H/W

GF(p^n)

- Definition of generating polynomial g(X) is parallel to the generator in Z_p:
  - every element in GF(p^n) (except 0) can be expressed as a power of g(X)
  - the smallest exponent k such that $g(X)^k \equiv 1$ is $p^n - 1$
- Discrete log problem in GF(p^n):
  - given h(X), find an integer k such that
    $$h(X) \equiv g(X)^k \pmod{P(X)}$$
  - believed to be very hard in most situations

Recursive GCD

```
int gcd(int p, int q) // assume p >= q
{
   int ans;

   if (p % q == 0)
      ans = q;
   else
      ans = gcd(q, p % q);

   return ans;
}
```

```
int gcd(int p, int q)
{
   int r = p%q;
   if (r == 0)
      return q;
   return gcd(q, r);
}
```

Recursive Extended GCD

- Given $a>b\geq 0$, find $g=GCD(a,b)$ and x, y s.t. $a\,x + b\,y = g$ where $|x|\leq b+1$ and $|y|\leq a+1$
- Let $a = q\,b + r$, $b>r\geq 0 \Rightarrow (q\,b + r)\,x + b\,y = g$
  $\Rightarrow b\,(q\,x + y) + r\,x = g$
  $\Rightarrow b\,y' + r\,x = g$, where $y' = q\,x + y$
- This means that if we can find y' and x satisfying $b\,y' + (a\%b)\,x = g$
  then x and $y = y' - q\,x = y' - (a/b)\,x$ satisfies $a\,x + b\,y = g$
  Note that in this way r will eventually be 0

```
void extgcd(int a, int b, int *g, int *x, int *y) { // a > b >=0
    if (b == 0)
        *g = a, *x = 1, *y = 0;
    else {
        extgcd(b, a%b, g, y, x);
        *y = *y - (a/b)*(*x);
    }
}
```